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Cissoids and Conchoids

by

Grace Madeline Pugh

A Thesis Submitted for the Degree of

Master of Arts

University of Wisconsin

1917

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Cissoids and Conchoids

The early second century B.C. gave us the names of three mathematicians, Hypsicles who added the fourteenth book to Euclid Elements, Diocles who invented the ivy shaped curve, the cissoid, and Nichomedes who invented the mussel-shaped curve, the conchoid. The date of Diocles can be determined by two facts only, that Geminus knew the cissoid by name, and that Diocles lived after Archimedes for he wrote a commentary on his work.(2) The work in which this occurs, also contains a definition of the cissoid. Let AOA' and BOB' be two fixed diameters of a circle at right angles to each other. Draw two chords QQ' and RR' parallel to BOB' and equidistant from it. Then the locus of the intersection of AR and QQ' will be a cissoid. Its equation can be expressed in the form $y^2(2a-x)=x^3$. This curve was used to duplicate the cube. For, if OA and OE are the two lines between which it is required to find the geometric means, and if A'E cuts the cissoid in P, and AP cuts OB in D, then OD-OA.OE. OD is one of the required means. (3)

Probably at the same time as Diocles Nichomedes invented a curve called the conchoid. The conchoid was known to Geminus in 70 B.C., and Eutocius says that Nichomedes made fun of Erastosthenes mesolabium and said that his own was superior. Probably Erastosthenes had not been dead a long time. The treatise in which Nicomedes discussed the conchoid is known to us

1.Ball,History of Mathematics p.78.

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from Eutocius' commentary on the duplication of the cube, from Pappus(4), and two or three remarks of Proclus. The conchoid is a curve such that the straight line joining any point on the curve with a given point is cut by a given straight line so that tha segment between the curve and the given straight line is of a given length. Nichmedes invented a little maching for describing it. AB can be moved horizontally on DE to which it is confined with a button,C sliding in a grove.(fig.1) AC is constant in length. Point E is called tha pole.

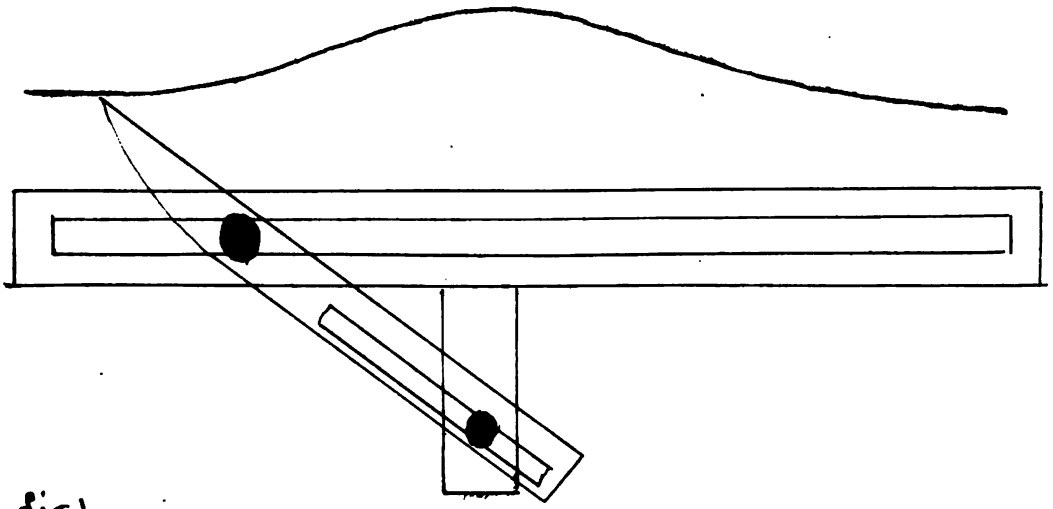

fig 1

The method of the duplication of the cube by means of thw conchoid is as follows. Let AD and AB (fig.2) be the given straight lines between which we want to find two mean proportionals. Place these at right angles and complete the rectangle

2.Gow, History of Mathematics p.269.

3.W.W.R.Ball, Short History of Mathematics p.85.

4.Cantor,Book 4,pp244-246.

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ABCD. Extend BC to N, making BN=BC, and join N to D, bisecting AB in L. Bisect BC in E and draw EF at right angles to BC, so that FC=BL. Join N to F, and through C draw CT parallel to NF.

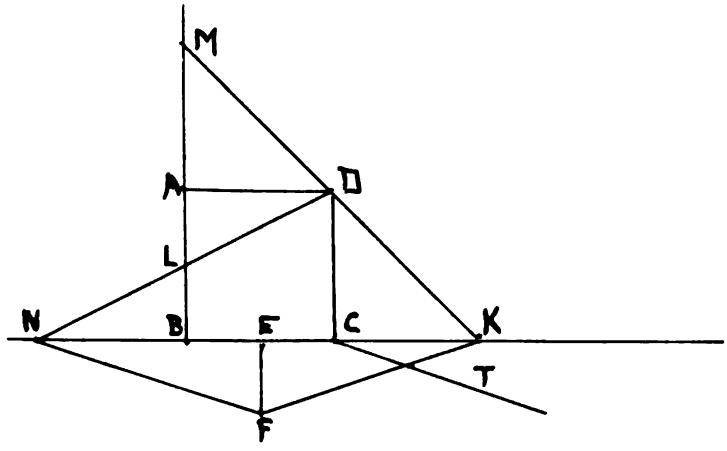

Fig.2

From F as pole with CT as fixed axis and BL as lenghh of the constant segment describe a conchoid cutting NB in K. Join K to D, and let KD intersect AB in M. Then AM and CK are the required mean proportionals. For by similar triangles

MA/AD=DC/CK,

therefore MA=AD.DC/CK;

andAD.DC=NC.BL.

Therefor MA=NC.TK/CK.

But NC/CK-FT/TK,

therefor MA=FT,

and $FK^2=KT-\frac{1}{2}AB=ML$.

By Euclidll 6, $FK^2=BK.KC-CF^2$,

and $ML^2=BM.MA-AL^2$,

therefor $BK.KC-CF^2=BM.MA-AL^2$.

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But CF=AL,

therefore BK.KC=BM.MA,

and BM:BK::KC:MA,

but BM:BK::CD:CK::AM:AD?,

therefore CD:CK::CK:AM::AM:AD.

The conchoid was also used to solve the problem of the trisection of a given angle. Proclus says Nicomedes solved the problem, but Pappus claims to have solved it. Let ABC be the angle to be trisected. From A draw AC perpendicular to BC. Complete the parallelogram. From B as a pole with AC as fixed line and 2AB as the constant distance describe a conchoid which shall meet LA produced in E. The line BE cuts AC in D. Bisect DE in N and join A to N. AN=NE=AB. Triangles ABN and ANE are isosceles. Therefore the exterior angle ANG=2AEN=2NBC, and ∠ANB=∠ABN=∠2NBC. (fig.3)

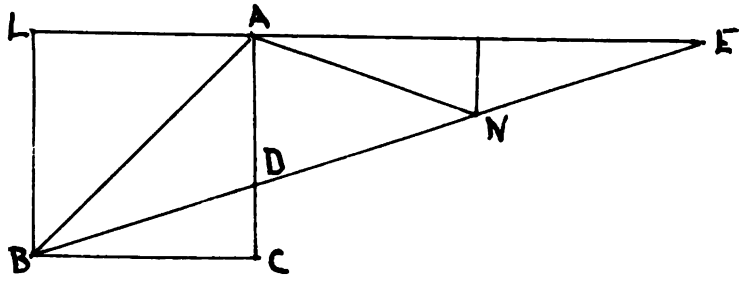

fig. 3

Inthis paper we shall study the cissoid and its special case the conchoid and the applications of these curves to geometry and mechanics.

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Chapter 2

Generalization

In the previous chapter we considered Diocles' definition of the cissoid. For the purpose of this paper we shall define the cissoid as a curve whose construction depends on two given curves T and T' and a fixed point O as a pole; the radius vector from O being the difference between the radii vectors from O to T and T'. We will consider the cases when T is a straight line or a circle, and T' a circle through O, also the case when T' is a circle with its center at O; the curves of this last construction are called conchoids.

To draw these curves the ancients ordinarily applied two methods, one geometry, the other kinematics. The geometrical method can be divided into plane and space curves. The curve will be found by construction from simple elements or as a section of a known surface, such as the conic section of the circular cone. In the kinematical method the motion of a point, line will be considered and defined, and the position of the element fixed.* To these we can add a third method which belongs to a later day, the analytical method. One gives to the defined function certain properties or requires the fulfillment of certain relations between given elements of the curve. These requirements, analytically formulated, lead to the algebraic equation of the curve. These methods are naturally not sharply separated.

* Wieleitner pp1,2.

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We will begin with a very simple construction by which a great family of curves can be drawn. We have given two curves T and T' and a point O in any position. Draw through O a line G which cuts T in P, T' in P'. On G mark off OQ=OP-OP' for all points P and P'. Move G about O so that Q describes a curve C which is called the cissoid of T and T' with respect to O. Then the cissoid is formed by the substraction of the radii vectors of the two given curves. Now join the finite intersections R of T and T', then the lines OR are tangent at O to the cissoid C. If T and T' are curves of the order m and n, C in general has a mn-triple point at O. Let a be the number of points of intersection at infinity. Let T go through O b times; T' through O d times. Under this hypothesis O is of order (mn-bd-a) for C. To determine the order of C, find the number of the points Q which lie on G. This number is as large as the number of combinations of the (m-b) points of intersection of P with T and the (n-d) points of intersection P' with T', or (m-b)(n-d). Just as we determine the order of the point P, we get the order of C. Thus we get the statement: the cissoid C of two curves T and T' of order m and n respectively which go through the pole b and d times and have a points of intersection at infinity is of order r=2mn-(bn+dm+a)

Thereis a simple method of drawing the tangent to C when the tangents of T and T' are known. Let T and T' be given in polar coordinates and the equations of the subnormals are:

(T) $S_n - d\rho/d\theta$

(T') $S'_n - d\rho_1/d\theta$-

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For C $\rho - \rho_2 - \rho_1$, and

$$S_m = d\rho/d\theta = \rho_2 - \rho_1' = S_m' - S_m.$$

The polar subnormal of the cissoid of two curves is equal to the difference of the polar subnormals of corresponding points of the two curves. Consequently the normal and tangent to C are easily constructed. The curvature of C is also known, if the curvature of T and T' are known. For then it is only necessary to be able to construct the radius of curvature R.

$$(1)\quad R = \frac{(\rho^2+\rho'^2)^{\frac{3}{2}}}{\rho^2+2\rho'^2-\rho\rho''}$$

$$(1')\quad \rho'' = \rho + 2\rho'^2/\rho - \frac{(\rho^2-\rho'^2)^{\frac{3}{2}}}{\rho R}$$

Let OP(fig4) $=\rho$, OM$=\rho'$, then MP$=(\rho^2-\rho'^2)=a$. Draw MB perpendicular to MP; then OB$-\rho'^2/\rho$, BP$-(\rho^2-\rho'^2)/\rho=b$. From (1')

$$(2)\quad \rho'' - \rho'^2/\rho = b(a-R/R).$$

CP$=$ R, CM'$=$ MC and M'A is parallel to CB, consequently AB$=b(a-R)/R$ and OA$=\rho''$.

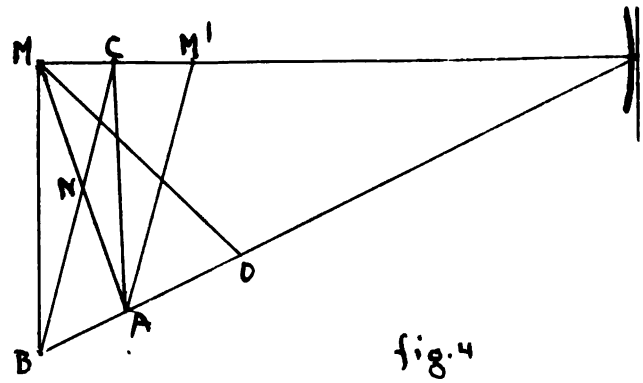

fig.4

If on the other hand ρ'' is given, then we need only to connect B with the middle point N of MA to obtain the center of curvature of C. For the cissoid $\rho''-\rho_2''-\rho_1''$, the radius of curvature R and R of T' and T can be constructed, and R can be constructed.

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The conchoid is merely a special case of the cisseid. Consider two planes K and K' placed one on another. In K let there be a fixed point O and a fixed curve T, in K' a line G and an it two fixed points P and Q at a distance l. Now move K' along K so that G always goes through O and P slides on K, then the point Q describes in K a conchoidm of K. And each point of G desetibes a conchoid of T. The defined sliding of the two plane along one another is called conchoidal.

The sliding of a plane on another plane when K is identical to K' must be considered. Such a construction is dependent upon three constants. We can assign to each point a given position and turn the planes about the point. Should the motion also be determined so that each point of K' describes a line or trajectory in K, then we have determined two conditions. In this way we can get in the simplest cases(1) the trajectory of two fixed points of K' in K, (2) the trajectory of a point and the envelope of a curve of K' in K, (3) the envelope of two curves of K' in K.

To understand the laws of such a sliding of one curve on another it is necessary to consider the motion as taking place discontinuously. We are given two points A and B of K'. Consider the initial positions A_0 and B_0 in K and a second position A_1B_1. (fig5) Consider the displacement of A_0 and of B_0. Draw A_0A_1 and B_0B_1 and construct perpendiculars to the mid points of thes lines. Their intersection M is the intersection of two concentric circles. A moves on a circle whose center is M , B moves on a circle whose center is M. Triangle A_0B_0M is similar to triangle

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$A_1 B_1 M_1$ as three sides are equal. Therefore angle x equals angle y and angle x plus angle a is equal to angle y plus angle a. Therefore the angular momentum is equal, and we can say that every displacement in the plane motion is a rotation. (Poinsot's theory of displacement.) The trajectories of all points of K' are concentric circles.

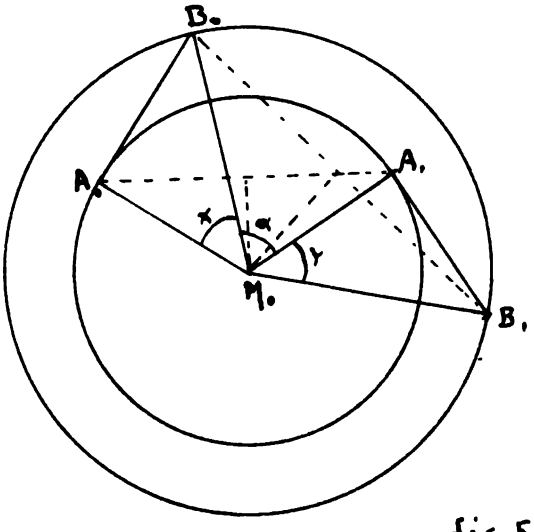

fig. 5

Consider the position of the points $A_0 B_0$, $A_1 B_1$ .... as infinitely close together so that A and B as well as all the other points describe a plane curve. Then the perpendiculars between two consecytive positions of A lies on the normal to the trajectory. The point M where these perpendiculars meet is called the instantaneous center.

A second property is stated as follows: any motion in a plane can always be generated by the rolling of one curve on another. Think of a body that undergoes a succession of displacements. Let the first displacement take place about C_1 as a center, the next about C_2, the third about C_3..... These are the points in space. C_1 is also a point in the body, the angular displacement

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is θ. C_2', C_3', C_4', (fig.6) are points in the body that become centers of rotation. Increase the num ber of sides of the polygon

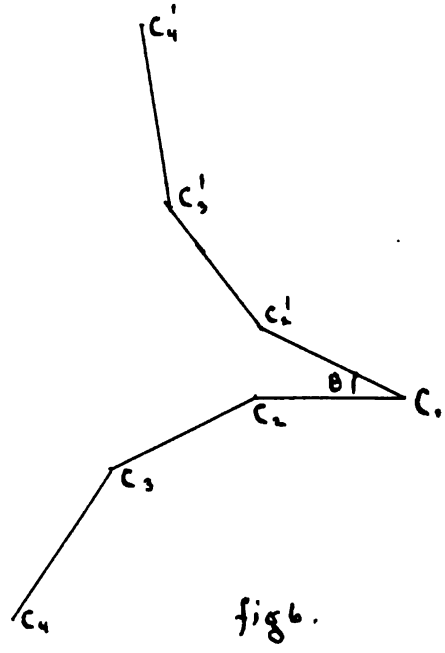

fig 6.

indefinitely and we get a smooth curve that is known as the pole curve. In mechands tha polecurve is usually known as the body centrode, the space curve as the plane centrode.

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Chapter 3

Special Cases of Cissoids

Pedal Curves of Conic Sections

To obtain special cissoids, we will take as the original curves two circles K and K' with radii r and R, with the pole O on K. Each circle has two points at infinity, and the order of C, according to the method in the previous chapter, is $2.2.2-(2+2)=4$. The centers M and M'(fig.7) have respectively the coordinates $(\frac{1}{2}p,\frac{1}{2}q)$ and $(m,0)$.

$OA=\frac{1}{2}p$

$MA=\frac{1}{2}q$

$OA\cos.\theta = OE$

$MA\sin.\theta=AB$

Therefore $OD=\frac{1}{2}\rho_1-\frac{1}{2}p\cos.\theta + \frac{1}{2}q\sin.\theta$,

therefore $(K)\rho_1 = p\cos\theta + q\sin\theta$.

Similarly for K',

$$\rho_2 - m\cos\theta \pm\sqrt{R^2 - m^2\sin^2\theta}.$$

Since the polar subnormal of two curves was shown in chapter 2 to be equal to the difference of the polar subnormals of the corresponding points of the two curves, we have $\rho = (m-p)\cos\theta - q\sin\theta \pm\sqrt{R^2 - M^2\sin^2\theta}$.

Let $\quad E\equiv x^2/a^2+y^2/b^2-1=0.$

This becomes the equation of an hyperbola if ib be substituted for b. Drop perpendiculars from O to all the tangents of E. The foot of these perpendiculars form the pedal curve F of E with respect to O. Let u be the eccentricity of the ellipse; then the equation of the tangents ~~are~~ is

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(1) $x\cos u/a + y\sin u/b - 1 = 0$

The equation of the perpendicular from O is

(2) $(x-\alpha)/b\sin u - (y-\beta)/a\cos u = 0.$

Eliminate u from (1) and (2), and we obtain

(3) $F \equiv (x^2+y^2-(\alpha x+\beta y))^2 - a^2(x-\alpha) - b^2(y-\beta) = 0,$

which is the equation of the desired pedal curve.

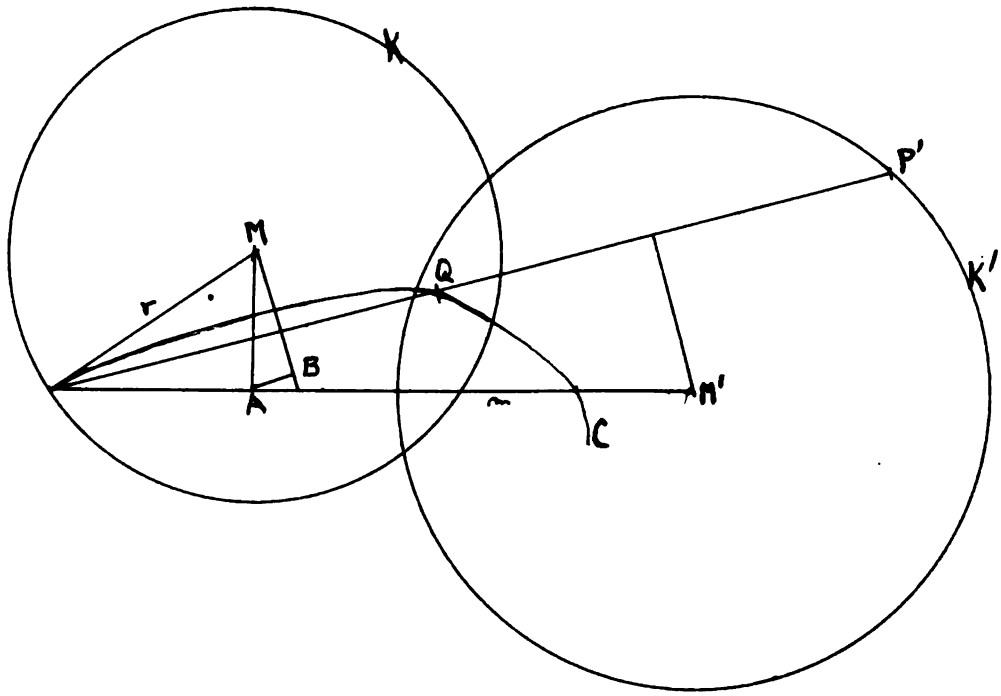

fig. 7.

Move the origin to O and the equation of F becomes

(3*) $(x^2+y^2) + 2(\alpha x+\beta y)(x^2+y^2) + (\alpha^2-a^2)x^2 + 2\alpha xy + (\beta^2-b^2)y^2 = 0.$

Transform to polar coordinates

(4) $\rho = -(\alpha\cos\theta - \beta\sin\theta) \pm \sqrt{a^2-(a^2-b^2)\sin^2\theta}.$

F is identified with C when we place $\alpha = p-m$; $\beta = q$; $a = R$; $b = \sqrt{R^2-m^2}$; $p = \alpha \pm e$; $q = \beta$; $m = \pm e$. Therefore each pedal curve of an ellipse or hyperbola can be considered a cissoid of two different circles with the pole on one of the circles.

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The pedal curve may also take the form of a bicircular rational quartic Q. This construction comes from the conic section E' by circular inversion (transformation of reciprocal radii) from the point O. Let A be a point of E', A' the conjugate on AO refered to the circle of inversion M. A' describes the quartic Q. T is the polar of A refered to M. Therefore T goes thrpugh A' and is perpendicular to OA. T touches the polar reciprocal cutting OE of E' refered to M. Then Q is the pedal curve of E. To generalize we obtain from figure 8 the statement; the pedal curve Q of any curve E refered to a pole O is the

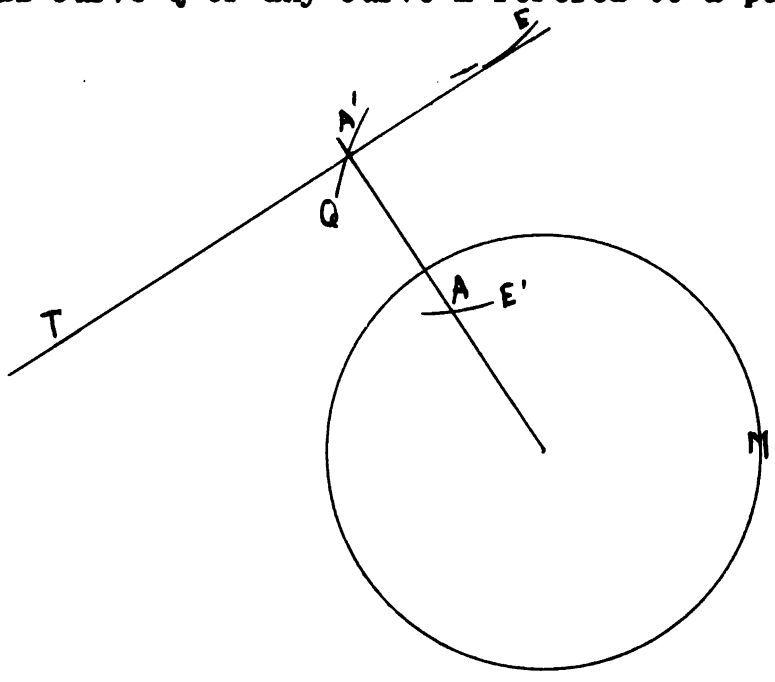

fig.8.

inverse of the polar reciprocalE' of E refered to a circle M with its center at O.

Therefore from figure 9 we get a direct geometrical explanation of the cissoidal construction of the pedal curves of conic sections. Let E be any curve. P describes the pedal

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curve of E refered to the pole F. O is another pole; Q and P are feet of perpendiculars to a tangent of E. Ao is perpendicular to FP. Therefore QO - FP - FA. A moves on the circle OF which cuts E. Therefore the pedal curve H with pole at O of a curve E is the cissoid of the pedal curve F with respect to the

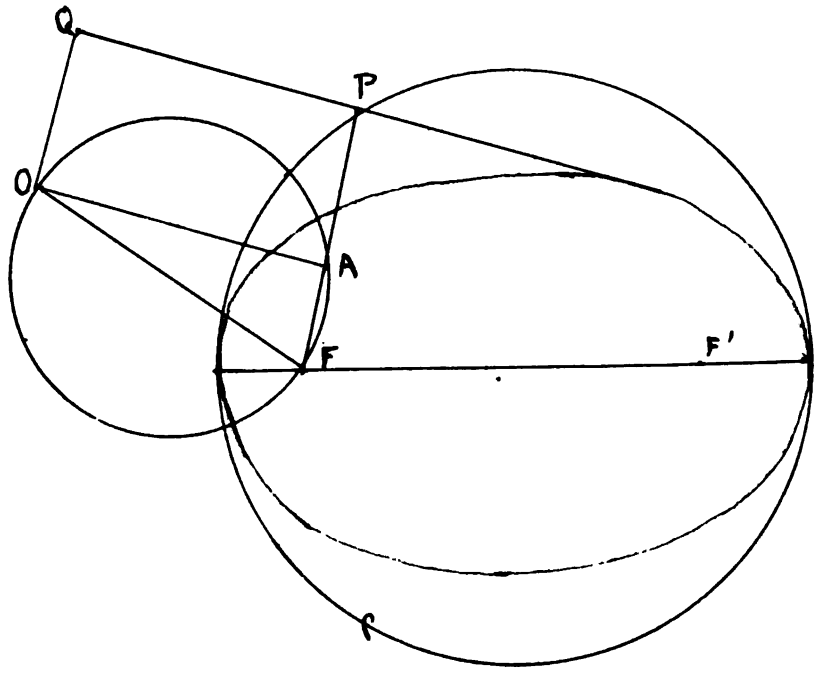

fig.9

pole F and cutting circle C. If E is a conic section, and F a focus,P is a point of the major auxiliary circle. And we have the cissoidal construction of H on two circles.

We can obtain by the geometrical method the pair of circles of the pole belonging to the conic section and vice versa. Let the pole inside one circle, and the pedal curve is an ellipse; if it lies out side, the pedal curve is an hyperbola. A study of figures 10 and 11 will make this clear. The circles are so chosen, that they cut each other in real points.

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The analytical meaning is that the discrimant D of the expression (- a)x - 2 xy - (- b)y is positive. Therefore D - ab. (/a - /b - 1). As a result O(,) does not lie inside the ellipse E. If the circles are tangent, there is a cusp inside the ellipse, outside the hyperbola.

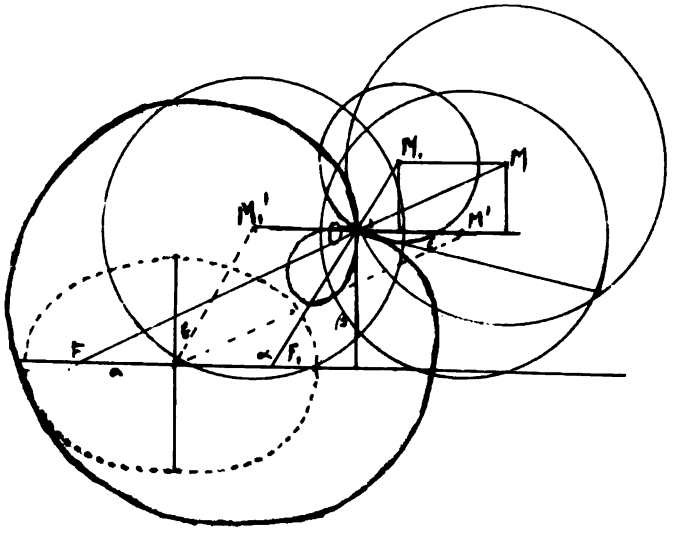

fig.10

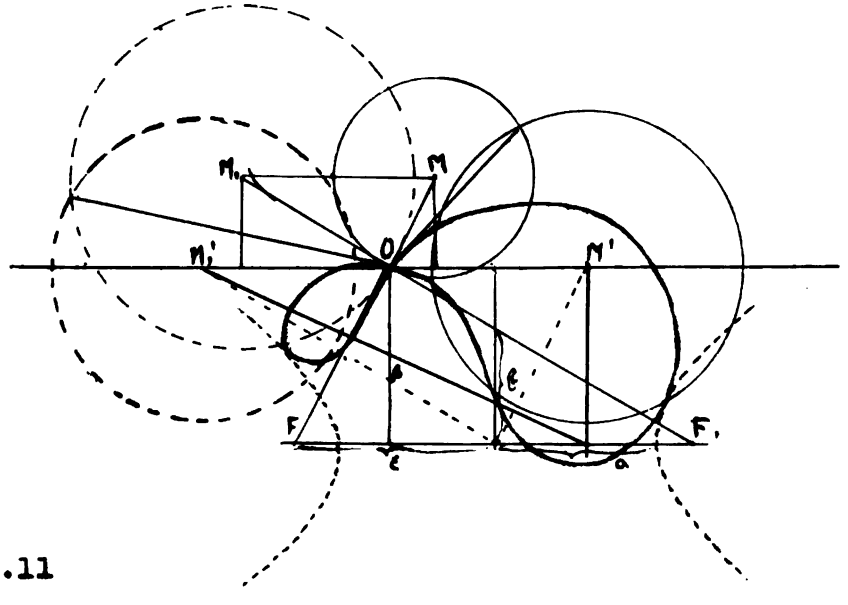

fig.11

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The Lemniscate of Booth and Bernoulli.

We get wellknown types of the pedal curves when (1) we give special positions to the pole, (2) when we take special conic sections as the base. Take the middle point of the conic section as the pole. The curve is then symmetrical to both axes and is called the lemniscate of Booth. Its equation is (- -0)

(1) $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

Then $p = m = \pm e$, $q = 0$, $R = a$, $r = \frac{1}{2}e$; and the center of one generating circle is at half the focal distance, the center of the other circle, M', is at the focus. (fig.12,13) The second of the pairs of circles is symmetrical to the conjugate axes. Take any line $OQ = OP' - OP$, and the middle point N of PQ from m has the constant distance $\frac{1}{2}P'M' = \frac{1}{2}a$. Draw a circle with its center at M and with thwe radius $\frac{1}{2}a$, and call the second point of intersection of ON with the circle N'. Then we get a Booth lemniscate, if we extend as radius vector the chord OP; then subtract the distance OP to obtain the point Q. As the radius vector is rotated, the point Q describes a Booth lemniscate.

The tangents of the double point O (fig,13) are given by the equation $a^2x^2 - b^2y^2 = 0$. They lie to the conjugate axes as the asymptotes $(x^2/a^2 - y^2/b^2 = 0)$ lie with reference to the hyperbola's major axes. They are imaginery in the case of the ellipse. Both the tangents at the double point $(ax \pm iby = 0)$ intersect in four coincident points, and the equation is of third order. To get the two other double points we must examine the imaginery circular points J and J', and transform thw equation (1) to

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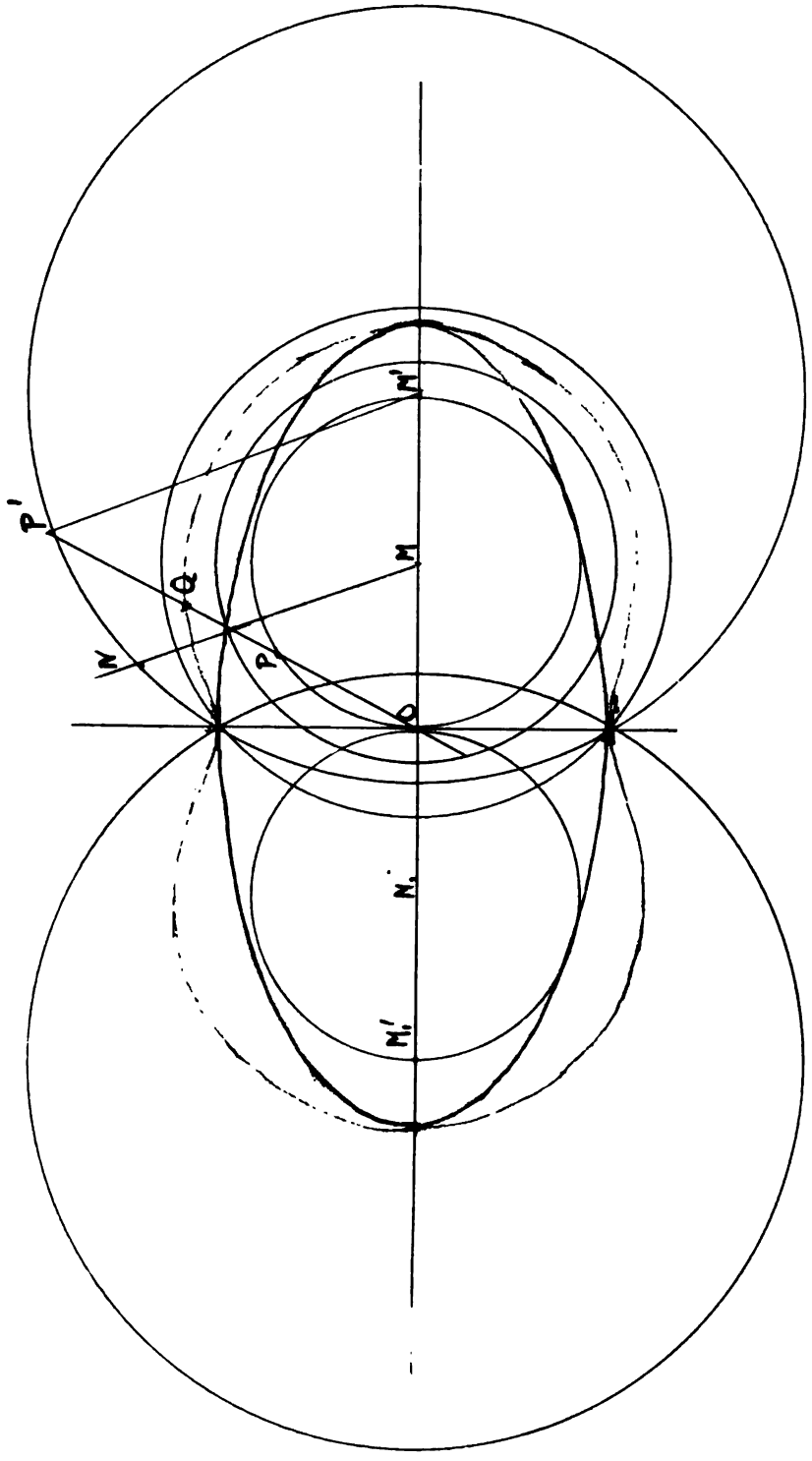

fig.12

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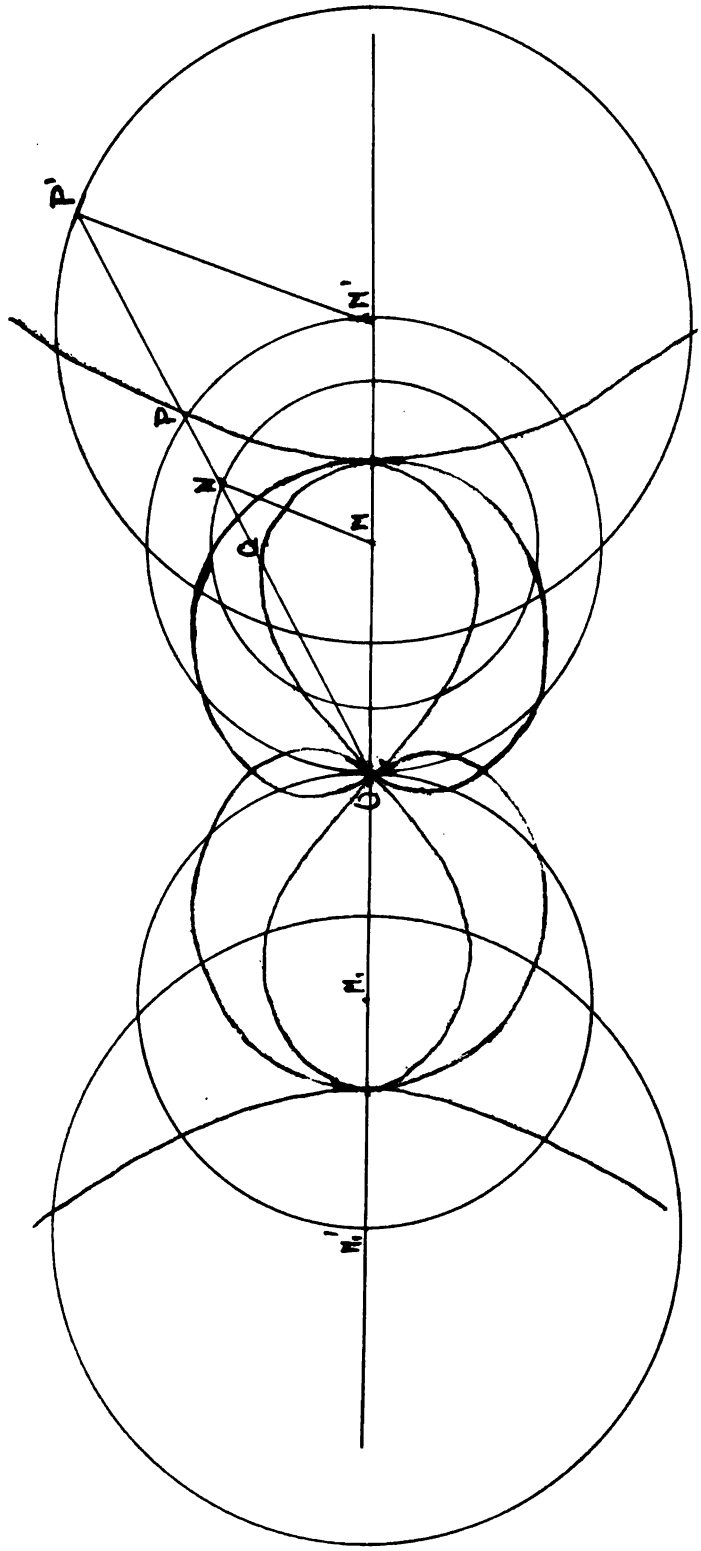

fig. 13

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the new triangle of reference OJJ'. Place

$$OJ = x + iy = \xi .$$

$$OJ' = x - iy = \eta .$$

$$JJ' = z = \zeta(-1).$$

Then equation (1) becomes

(1*) $$4\xi^2\eta^2 e\left(a^2(\xi + \eta) + b^2\zeta^2\left(\xi^2 - \eta^2\right)^2\right) = 0.$$

The highest powers of and give the double points at J and J'. These are 4 - (a - b) - 0 and 4 - (a - b) - 0 or

(2) $$\xi = x + iy = \pm\tfrac{1}{2}e$$

$$\eta = x - iy - \pm\tfrac{1}{2}e.$$

$\xi = 0$ and $\eta = 0$ when $a^2 = b^2$. Dividing (1) by the factor $(x^2 + y^2)$, and placing $\xi = \pm\tfrac{1}{2}e$ in (1*), we get the factor ζ three times. To get the four points of intersection of the tangents of the double points with the curve, the equation is $e(a^2 - b^2)\zeta + 4(a^2 + b^2) = 0$, which is satisfied when $\zeta = 0$ and when $a^2 = -b^2$. The pedal curve of the equilateral hyperbola with the equation

(3) $$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$

or in polar coordinates

(3*) $$\rho = a\sqrt{\cos 2\theta}$$

has three points of inflection, two of which are at the imaginery circular points, This is known as the Bernoulli lemniscate. (fig. 13). The Bernoulli lemniscate belongs to the family of Cassini ovals.*

* Salmon ,Higher Plane Curves ,p.42.

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Quartics with Three Points of Inflection

The Bernoulli lemniscate is related to all curves of fourth order which have three points of inflection. The family of curves has a group of common properties. These quartics are projectively related. Let the triangle of inflection O_1, O_2, O_3, be the fundamental triangle for the homogeneous point coordinates x_1, x_2, x_3, line coordinates u_1, u_2, u_3. Draw a polar triangle of a conic section H, whose equations in line and in point coordinates are

(1) $$H = \alpha_1 u_1^2 + \alpha_2 u_2^2 + \alpha_3 u_3^2 = 0,$$

$$H = \alpha_1^{-1} x_1^2 + \alpha_2^{-1} x_2^2 + \alpha_3 x_3^2 = 0.$$

Draw the pole P of the fundamental triangle to each tangent T of H. This pole describes a curve of the desired kind.

If T has the coordinates v_1, v_2, v_3, and if we use the fundamental triangle as a curve of third order with the equation $u_1 u_2 u_3 = 0$, then the coordinates of P are

(2) $\theta x_1 = v_2 v_3$; $\theta x_2 = v_3 v_1$; $\theta x_3 = v_1 v_2$.

From (2) we get

(3) $$v_1^2 = \theta x_1^{-1} x_2 x_3; \quad v_2^2 = \theta x_1 x_2^{-1} x_3; \quad v_3^2 = \theta x_1 x_2 x_3^{-1}.$$

Place these values in (1), we get the equation of the curve described by P,

(4) $$Q = \alpha_1 x_2^2 x_3^2 - \alpha_2 x_3^2 x_1^2 - \alpha_3 x_1^2 x_2^2 = 0$$

The curve Q has at the vertex O adouble point with a pair of tangents $\alpha_1 x_2^2 + \alpha_2 x_3^2 = 0$ which cut the curve in no other point. Each tangent is a tangent at a point of inflection. The same is true for the other vertices The equation of the curve Q, if placed in the form

(4*) $$Q \equiv \alpha_1 x_1^{-2} + \alpha_2 x_2^{-2} + \alpha_3 x_3^{-2} = 0.$$

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becomes by means of the quadratic transformation the conic section, $H' = \alpha_1 x_1^{-2} + \alpha_2 x_2^{-2} + \alpha_3 x_3^{-2} = 0$. This conic section H' which has the fundamental triangle for the polar triangle is the polar reciprocal of H in relation to a third conic section X of the kind whose equation is $x_1^2 + x_2^2 + x_3^2 = 0$. The curve Q has four double tangents.*

If Q is real, the fundamental triangle is real in order that the coefficients should be real and not all of the same sign. But only two points of inflection can have real branches. The third is necessarily isolated.

Let $x_1 = x = 0$, $x_2 = y = 0$ be two perpendicular axes, and let $x - z = 0$ be the infinitely distant line. Let the conic section be an ellipse

(5) $$H \equiv x^2/a^2 + y^2/b^2 - 1 = 0,$$

such that

$$\alpha_1 = a^2, \ \alpha_2 = b^2, \ \alpha_3 = -1.$$

Draw a tangent T from H to the axes cutting at p,q, so that the pole P has the coordinates -p,-q. The generated curve is called the "cross curve." Its equation is

(6) $$x^2y^2 = a^2y^2 + b^2x^2.$$

This has as the construction shows(fig.14) an infinitely distant double point and the asymptote $x = \pm a$, $y = \pm b$.

Take the same axes, but use the hyperbola

(7) $$H \equiv x^2/a^2 - y^2/b^2 - 1 = 0$$

as the fundamental conic, and we obtain the " Kohlenspitze" curve.(fig.15) It has an infinitely distant double point and

*Wieleitner "Spezielle Ebene Kurven" p 18.

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asymptotes $x = \pm a$. At the origin the asymptotes of the hyperbola are tangents. Its equation is

(8) $$x^2y^2 = a^2y^2 - b^2x^2.$$

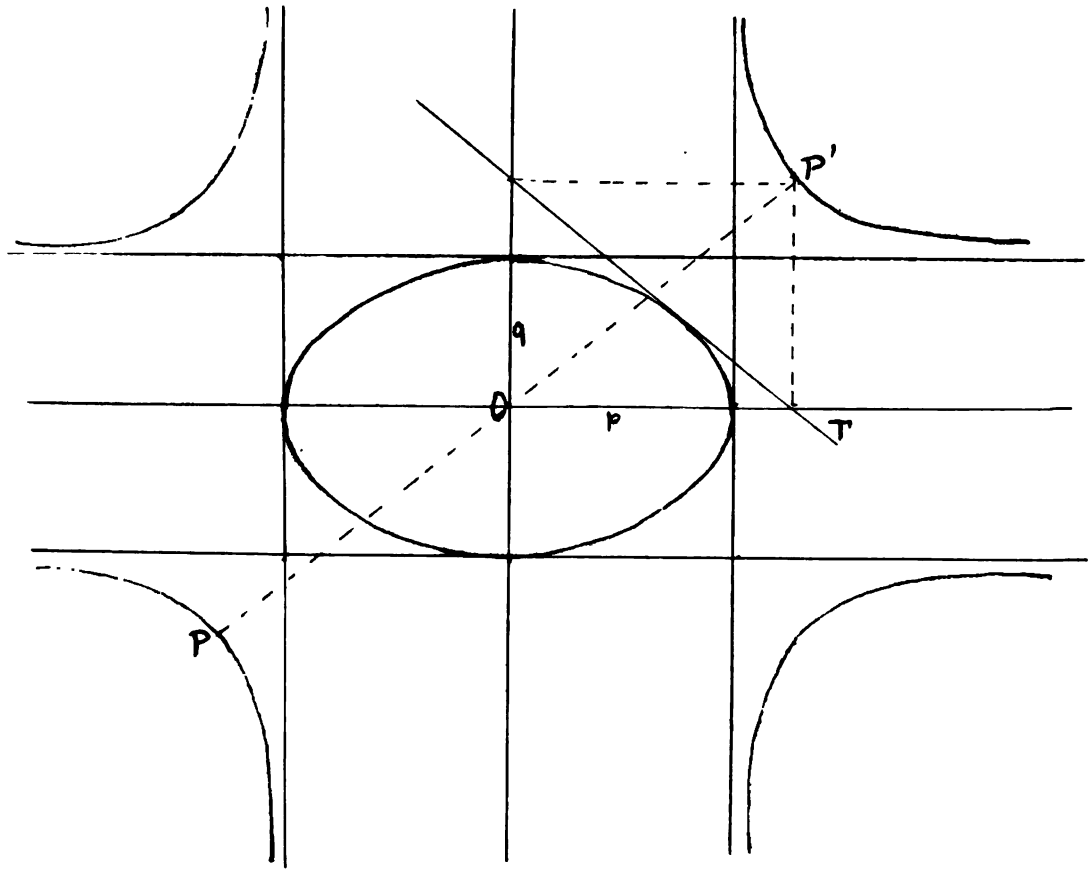

Fig. 14

Through the intersections of the tangents to a conic section with the axes construct a parallelogram whose sides are parallel to the axes. The vertex P' of the parallelogram describes a cross curve if the conic is an ellipse, a Kohlenspitze curve if the conic is an hyperbola.

If we take the lines $x = 0$, $y = 0$, $y = \pm b$ as the sides of the polar triangle, and use the hyperbola $x^2/a^2 - y^2/b^2 = 1$ as the conic section, the equation of the hyperbola becomes

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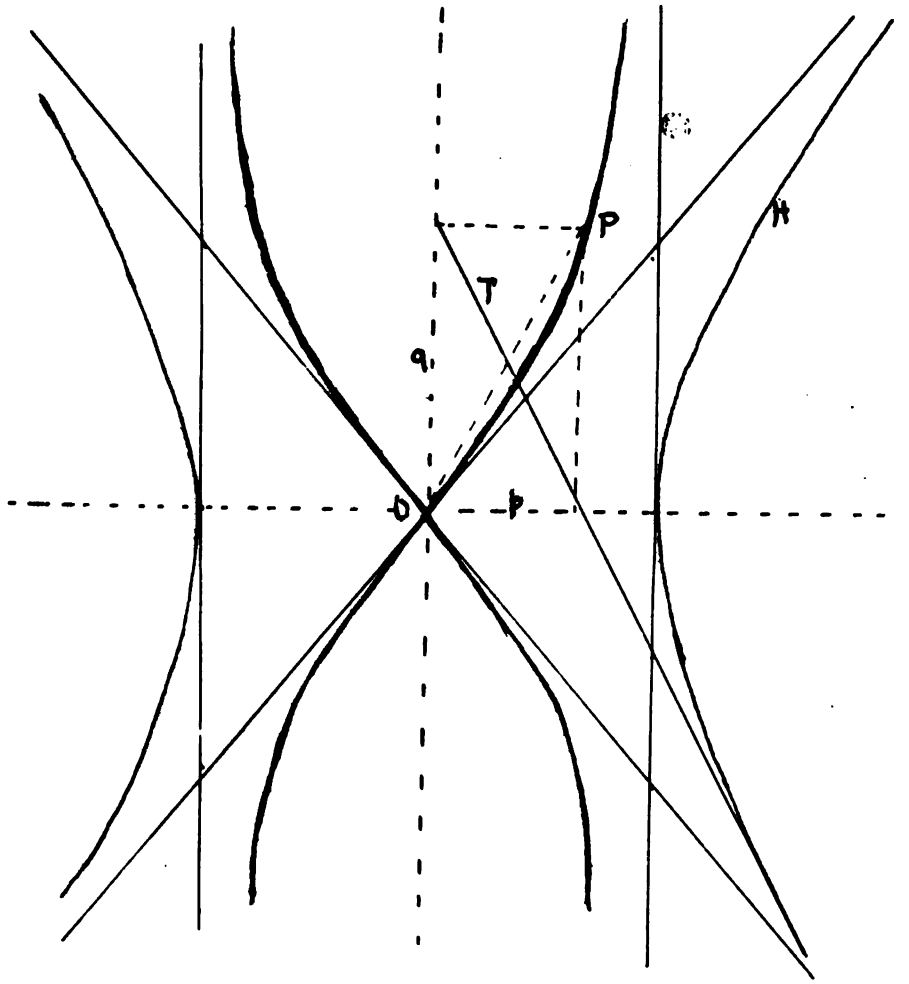
(9) $x^2/a^2 - 1/2b^2(y+b)^2 - 1/2b^2(y-b)^2 = 0$

such that $\alpha_1 = a^2$, $\alpha_2 = \alpha_3 = -2b^2$. The quartic will be

(10) $a^2(y^2 - b^2) - 4b^2x^2(y^2 + b^2) = 0.$



fig,15.

The construction of this curve which is called the hour glass curve is made as follows: given an hyperbola and the polar triangle, x - 0, y - b, y - -b. Any tangent to the hyperbola forms with the two parallel lines and the intersecting

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axis a trapezoid. The intersection P' of the diagonals of the trapezoid describes the "hour glass" curve. Fig,16.

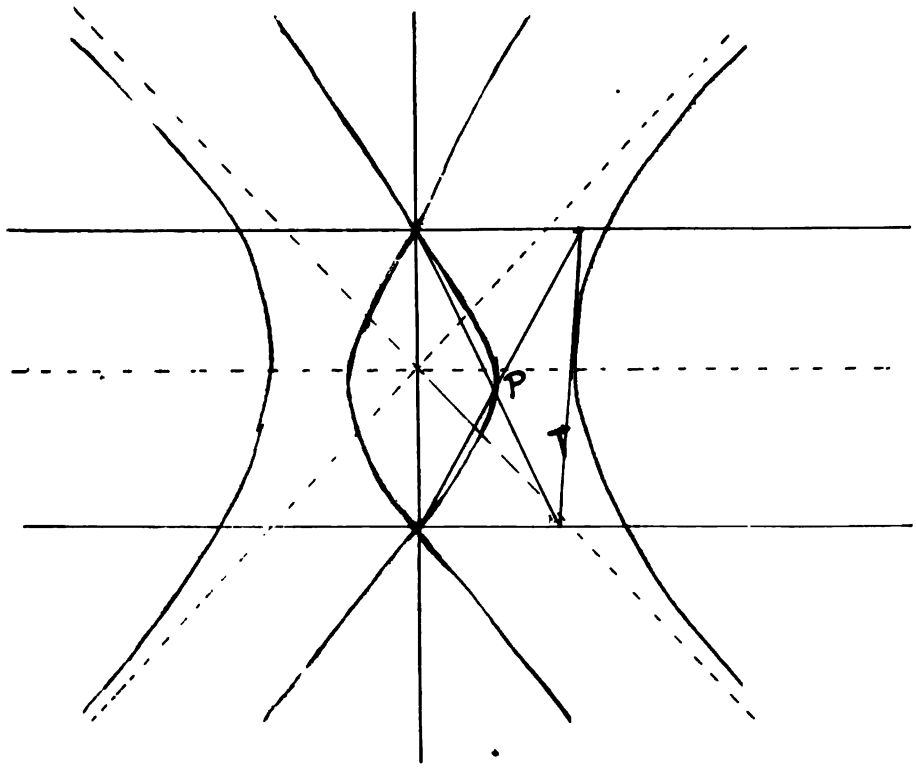

fig.16.

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The Pedal Curves of the Parabola.

We can obtain the pedal curve of the parabola by the construction of the pedal curves given in chapter two, if we use a straight line G instead of the circle K' with the radius R on which the pole does not lie. We may consider G a circle of infinite radius. Let $m - R = k$(fig. 7), and let m become infinite as R becomes infinite. Then the equation of the circle K' is

(1) $\rho_2 = m\cos\theta \pm\sqrt{R^2 - m^2\sin^2\theta}$.

If we extract the root, placing $m = k + R$, and let R become infinite, the equation of the line G at the distance k from O is

(2) $\rho_2 = k/\cos\theta$.

The equation of the circle K is

(3) $\rho_1 = p\cos\theta + q\sin\theta$.

From these equations we get the polar equation of the cisseid of the circle and the line with the pole on the circle.

(4) $\rho_2 - \rho_1 = \rho = k/\cos\theta - p\cos\theta - q\sin\theta$,

or in rectangular coordinates

(4*) $x(x^2 + y^2) = (k - p)x^2 - qxy - ky^2$.

This is the common circular cubic with double point at O.*

To determine the conic section to which this belongs, let $\alpha = -\infty$, $\beta = q$, $a = \infty$, $b = \infty$, $a + \alpha = p - k$, $\alpha + e = p$. This is a parabola with the vertex $S(x = k - p, y = -q)$ and the focus $F(x = -p, y = -q)$, where k is the distance from the vertex to the focus (fig.17). Its equation is

(5) $(y + q)^2 + 4k(x + p - k) = 0$.

* Wie leitner p 36.

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The cubic (4*) has the line G ($x = k$) as an asymptote. This cuts the curve in a point H ($y = -kp/a$) which is called the principal point. The real intersection of the other asymptotes, the focus B of the curve, is $x = -\frac{1}{2}p$, $y = -\frac{1}{2}q$. B is the middle point of the line between O and the focus F of the parabola; OH is the tangent to K and is perpendicular to BO. We can write

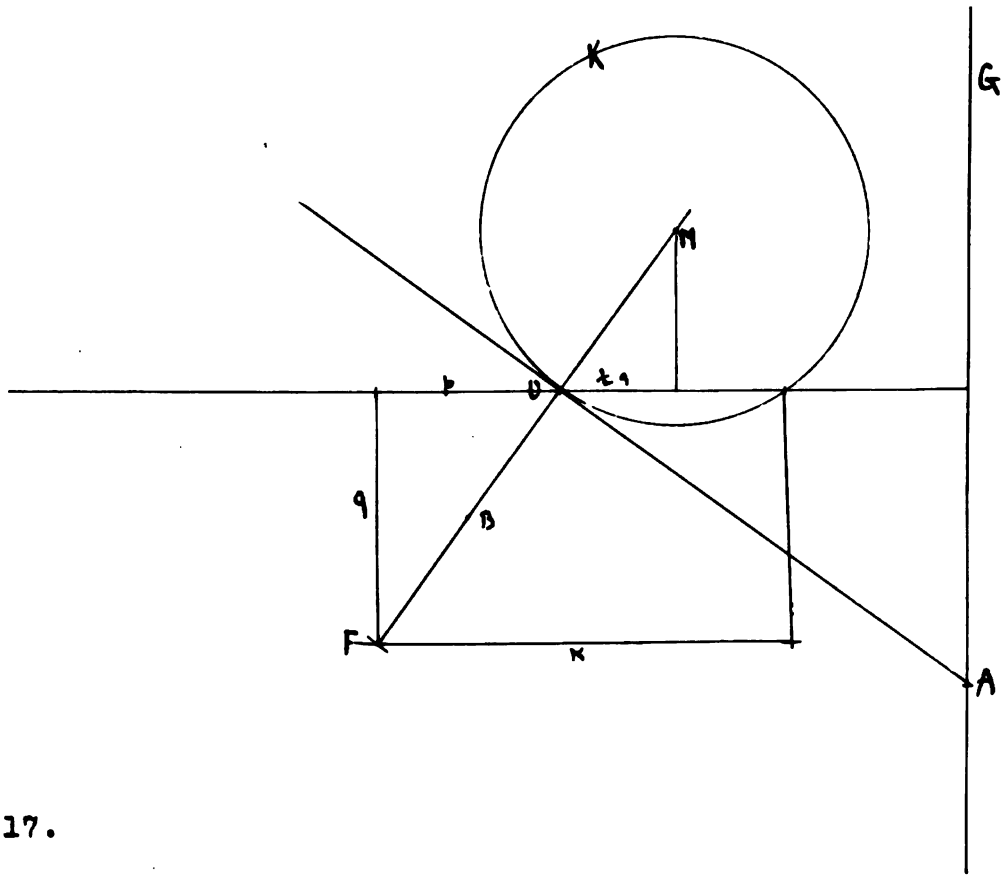

fig. 17.

(4*) in the form

(4#) $$(x - k)((x + \tfrac{1}{2}p)^2 + (y + \tfrac{1}{2}q)^2) + (kp - r^2)x + kqy + kr^2 = 0.$$

Draw a circle, $(x + \frac{1}{2}p) + (y + \frac{1}{2}q) = L$, about B as center and we get through the two finite points of intersection of this circle with the cubic a line $L(x - k) + (kp - \frac{1}{4}p^2 - \frac{1}{4}q^2)x + kqy + \frac{1}{4}k(p^2 + q^2) = 0$, which is tangent to the circle and goes through H. In this way we get a sheaf of concentric circles about B and a projective

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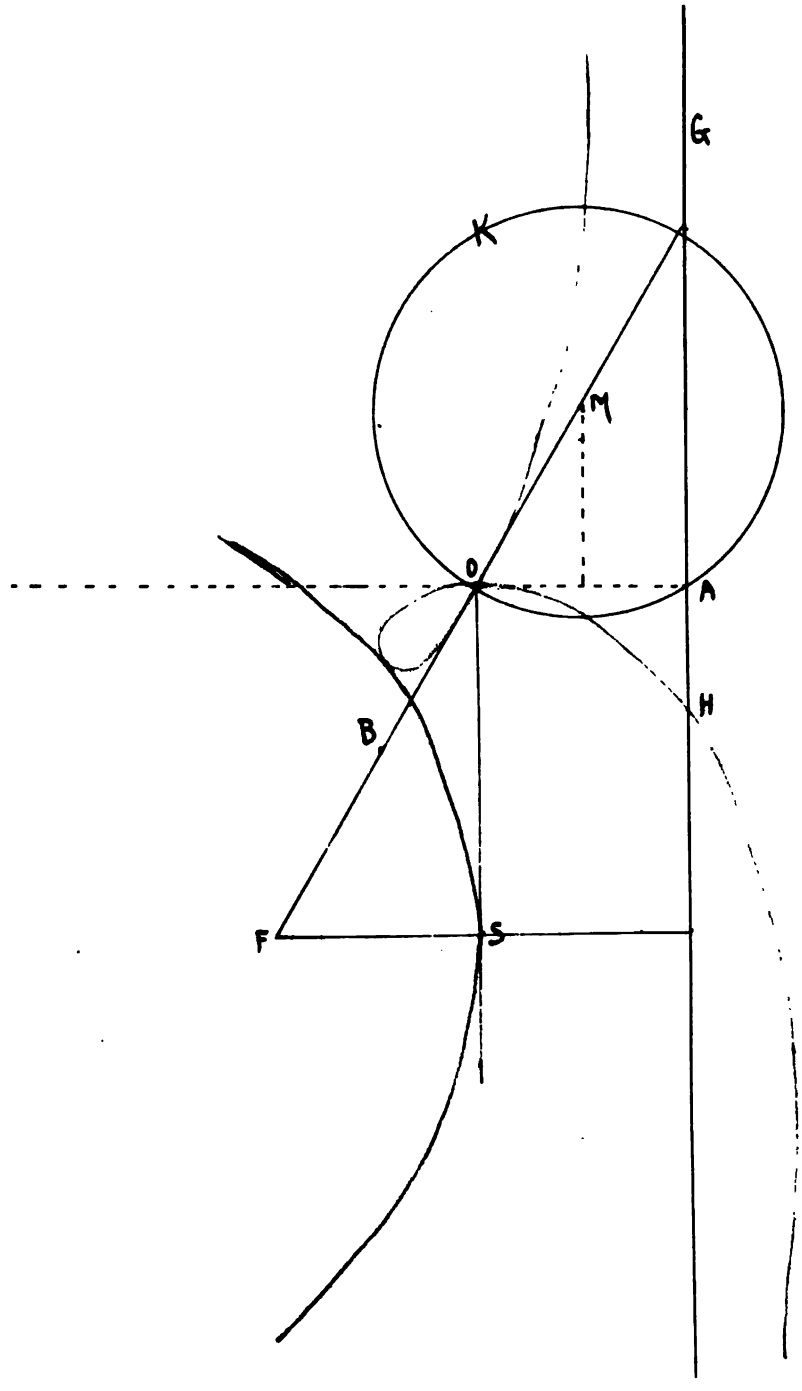

fig.18.

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sheaf of lines through H.

The first special case of the perpendicular of G and K is this: the foot of the perpendicular from O to G lies on K. The resulting circular cubic has the characteristic that a tangent at the double point is perpendicular to the asymptote. The vertex of the parabola lies on the line x - k - p - O. The curve is the pedal curve of the parbola from a point on the tangent to the vertex, and is called the Ophiuride. (fig.18) If G becomes tangent to K, so that OA is the diameter of K, O is the vertex of the parabola, and the Ophiuride becomes the cissoid of Diocles (fig.19).

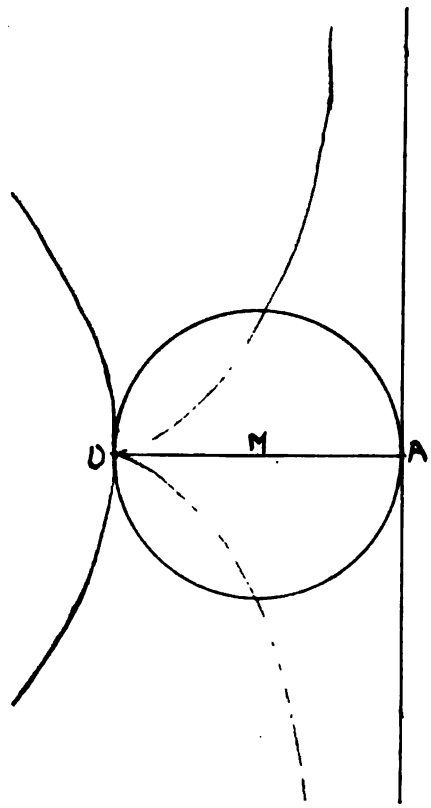

fig.19

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Another special position is for G to go through the center M of K. The curve is then specialized so that the tangents at its double point are perpendicular to each other. It is called the Strophoide. The focus B lies on the curve and O lies on the directrix of the parabola (fig. 20). Draw through B a line cutting the curve in A and A'. The intersections of OA, OA' with K are C and C', with G,D and D'. It is evident that triangle OAB is similar to triangle CDM, and triangle OA'B is similar to C'D'M. Therefore CMC' is a straight line and OA is perpendicular to OA'. Let N be the intersection of AA' with D. AN = A'N. We can construct the curve as follows. Given an angleBOE. Take any line through B which cuts OE in N. NA = NA' = NO. A and A' describe the strophoide. The strophoide is symmetrical with respect to the line BM (fig.21) when OMis perpendicular to G and is called the right strophoide. It is the pedal curve of a parabola for the intersection of the directrix and the axis. Its equation is

($\rho = 2k = 2r$; $q = 0$)

(6) $$x(x^2 + y^2) = r(y^2 - x^2).$$

Another special case of the position of G and K is to have G tangent to the circle K. In this case the curve has a cusp at O, and the parabola goes through O. Expressed geometrically $K - \frac{1}{2}p - \frac{1}{2}p - q$. The equation of the curve is

(7) $$x(x^2 + y^2) = k(y - xq/2k)^2.$$

The curve is called the oblique cissoid (fig.22) as opposed to the cissoid of Diocles which is often called simply cissoid. The equation of the cissoid of Diocles is

(8) $$x(x^2 + y^2) = 2ry^2.$$

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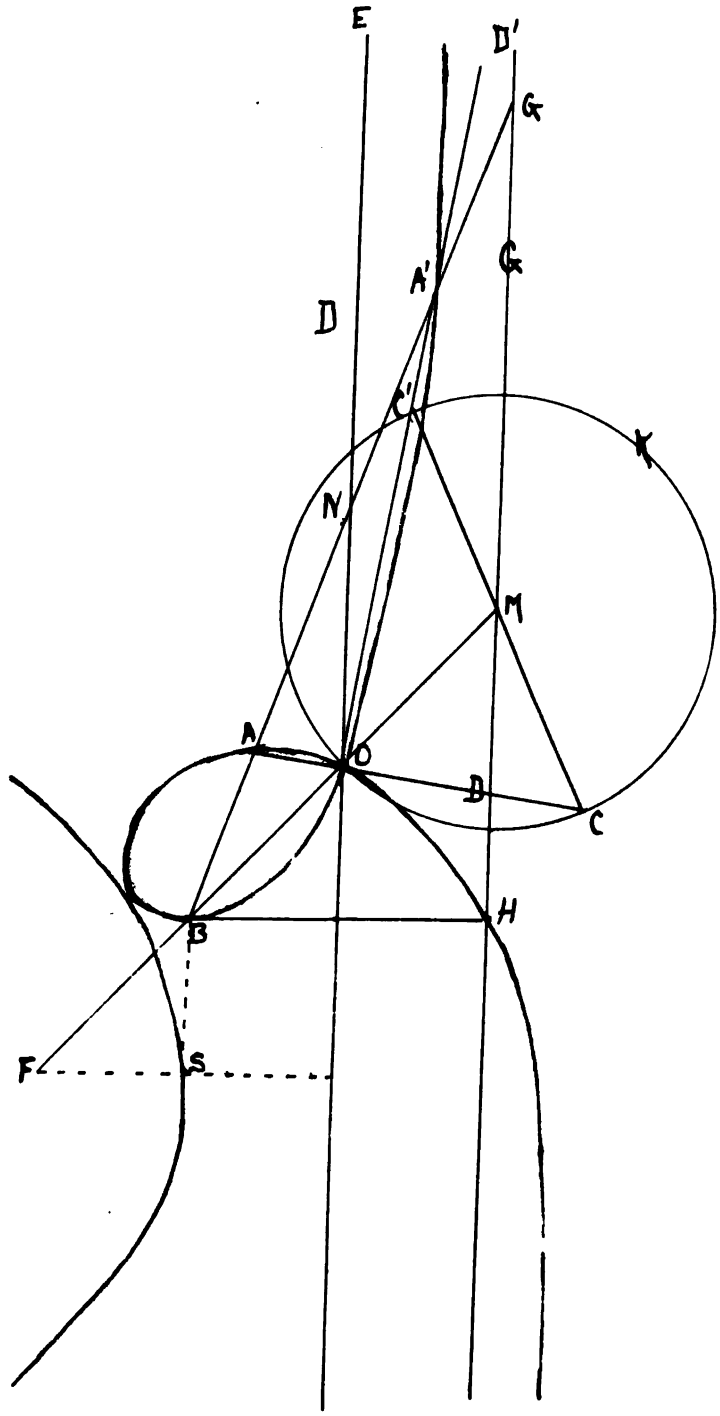

fig.20.

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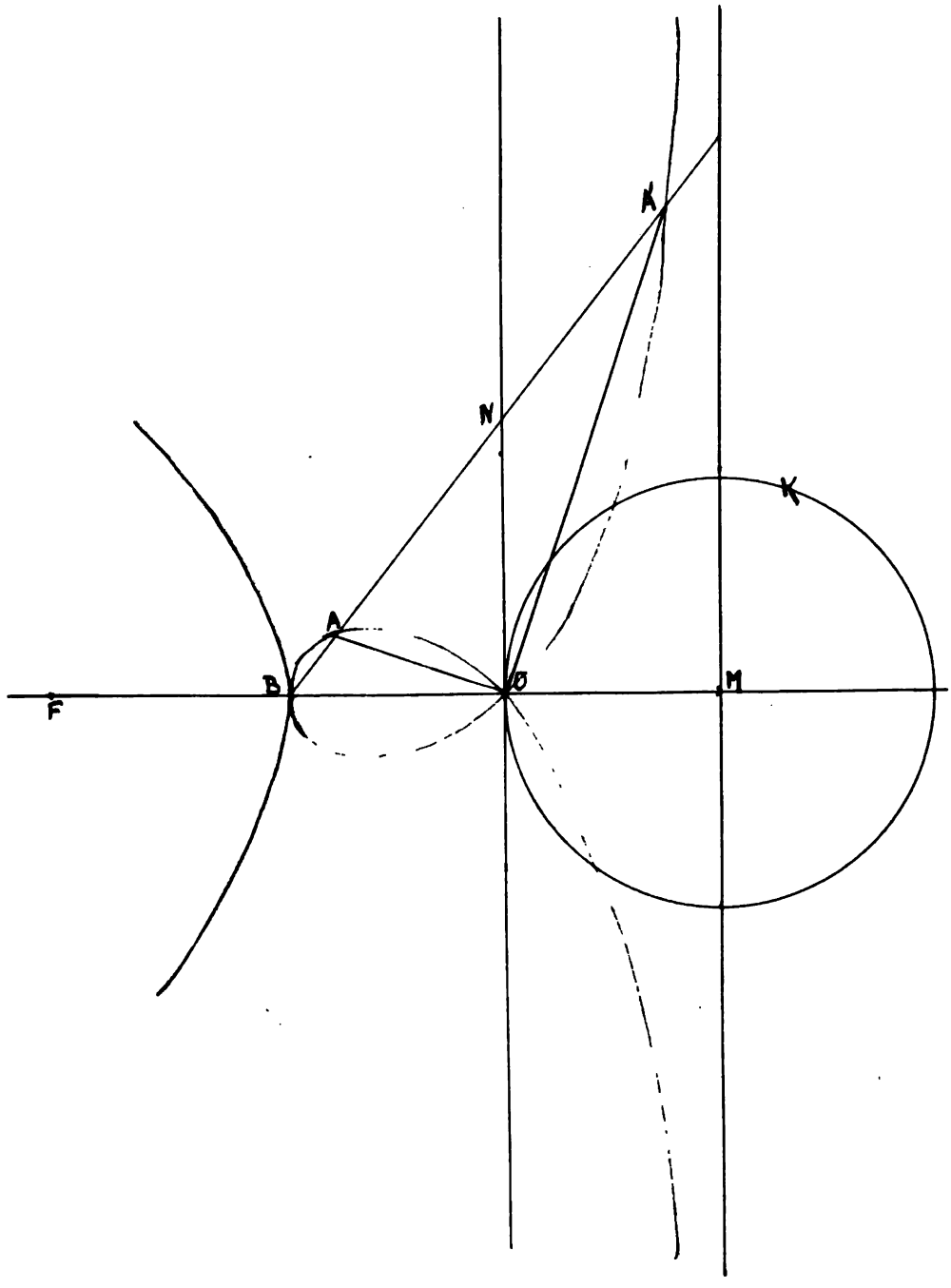

fig.21.

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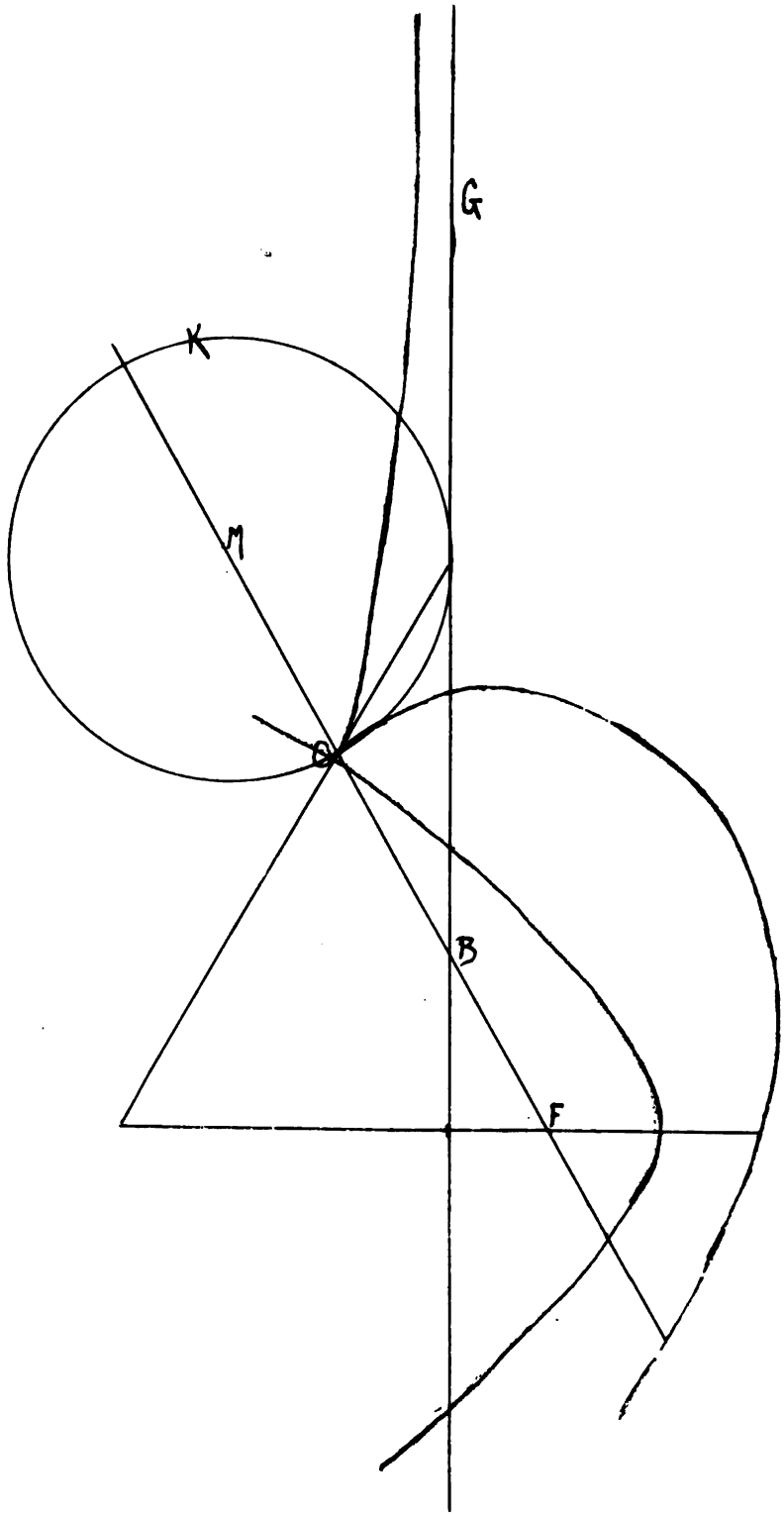

fig.22.

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In equation (4#) interchange the coefficients of x and y, and we get the expressions $kp = r^2$, and $q = 0$. The curve will have the equation

(9) $$2x(x^2 + y^2) = r(y^2 - 3x^2),$$

where $p = 2r$ and $k = \frac{1}{2}r$. The tangent at the double point makes an angle of 120 degrees. The curve is called the trisectrix of Mac Laurin(fig23). Transform (9) about B as center, and we get

(9*) $$(2x - 3y)(x^2 + y^2) + r^3 = 0;$$

or in polar coordinates

(9#) $$2\rho^3\cos\theta + 3\rho^2 r - r^3 = 0.$$

But $\cos\theta = 4\cos^3\frac{1}{3}\theta - 3\cos\frac{1}{3}\theta$, and we can write (9#) as follows: $8\rho^3\cos\frac{1}{3}\theta - 6\rho^3\cos\frac{1}{3}\theta + 3\rho^2 r - r^3 = 0$. This has the factor $(2\rho\cos\frac{1}{3}\theta - r)$. and equation (9#) becomes

(9") $$-\frac{1}{2}r/\cos\frac{1}{3}\theta.$$

With this curve we can trisect a desired angle θ. Place the apex of the angle at B, so that one side goes through the vertex S of the cubic and the other cuts the cubic in A. Erect at S the perpendicular to the polar axis and mark off on this perpendicular AS such that $BA' = BA = \rho$. Since $BS = \frac{1}{2}r$, $BA' = \rho = \frac{1}{2}r/\cos SBA'$ therefore angle $SBA' = \frac{1}{3}\theta$. The trisectrix of MacLaurin is the pedal curve of a parabola whose focus is $\frac{1}{2}r$ to the left of S, and whose directrix goes through B, and B is as far from O as the focus is from the directrix.*

*Wieleitner p 44.

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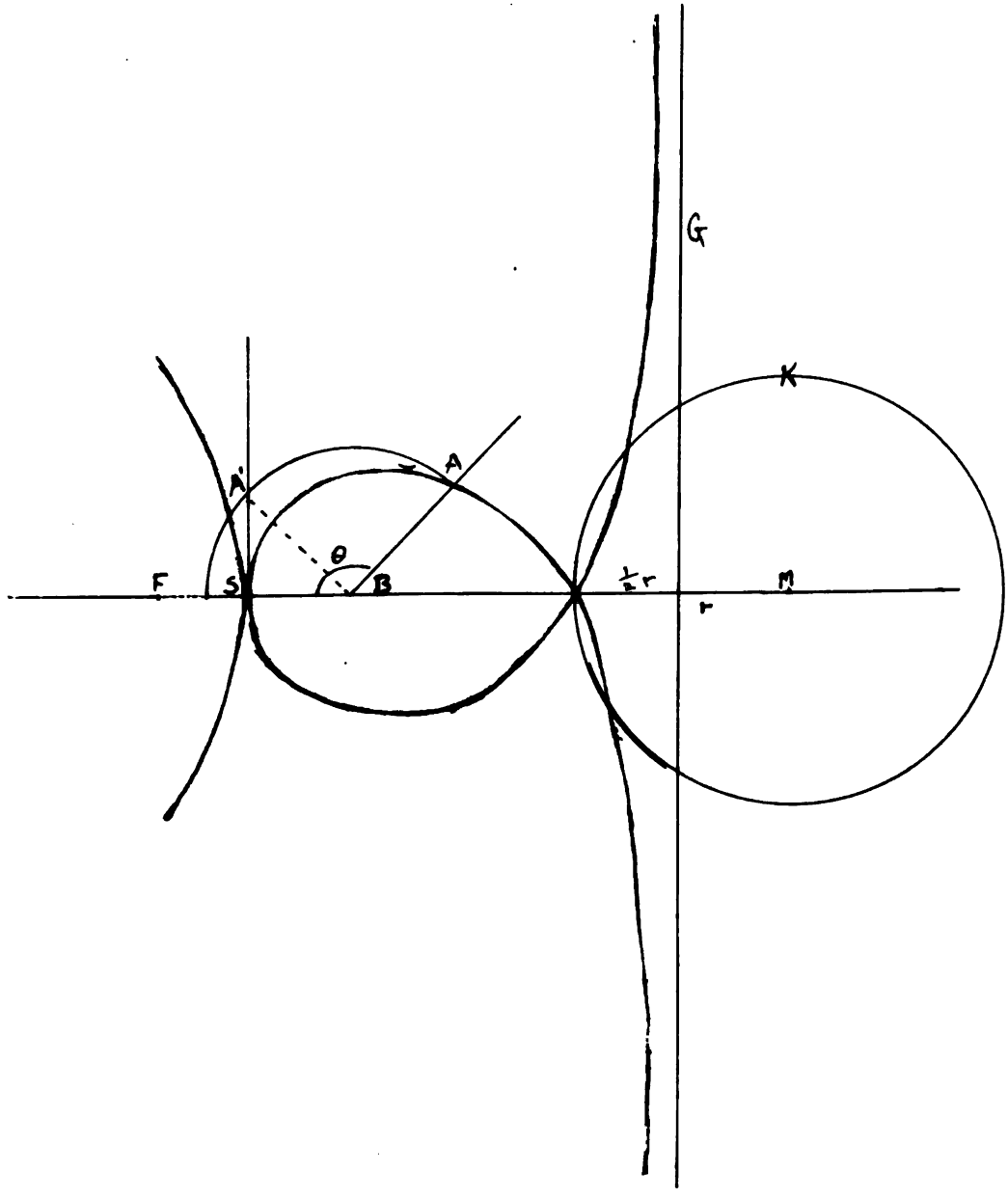

fig.23.

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Non-circular Rational Cubics as Cissoids.

We have considered the cissoids whose base curves were either circles or straight lines. Now we will consider the case when the base curves are a conic section G' and a straight line G, the pole lying on G'. The equations are

(1) $G \equiv x - k = 0,$

(2) $G' \equiv (y - ex + a)(y - mx + b) - ab = 0;$

or in polar coordinates

(1*) $\rho_1 = k/\cos\theta$

(2*) $\rho_2 = -((a + b)\sin\theta - (am + be)\cos\theta)/(\sin\theta - e\cos\theta)$
$(\sin\theta - m\cos\theta).$

The equation of the cissoid of G and G' is

(3) $\rho = \rho_1 - \rho_2 = k/\cos\theta + ((a+b)\sin\theta - (am+be)\cos\theta/$
$(\sin\theta - e\cos\theta)(\sin\theta - m\cos\theta);$

or in rectangular coordinates

(3*) $(x-k)(y-ex)(y-mx) = (a+b)xy - (am+be)x.$

This is an equation of a cubic with a double point at O. The line G is itself an asymptote; the other asymptotes are $y - ex - a = 0$ and $y - mx - b = 0$.

We will consider three cases , G as an hyperbola, parabola, and ellipse. The curves are all symmetrical; the pole is at the vertex of G'. The equation of the conic section has the form

(4) $y^2 = 2px + qx^2,$

for the ellipse $p = -b^2/a$, $q = -b^2/a^2$, $a = p/q$, $b^2 = -p^2/q$, $q + 1 = c^2/a^2 = e^2$. For the hyperbola $b = ib$; for the parabola $q = 0$. G keeps the same equation. The equation of the cissoid

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becomes

(5) $$(x - k)(y^2 - qx^2) + 2px^2 = 0.$$

In the first case we define the curve as follows: given a circle with radius OA = r; from A strike two arcs such that AC = 2AB = 2rw where w is the central angle of AB(fig.24).

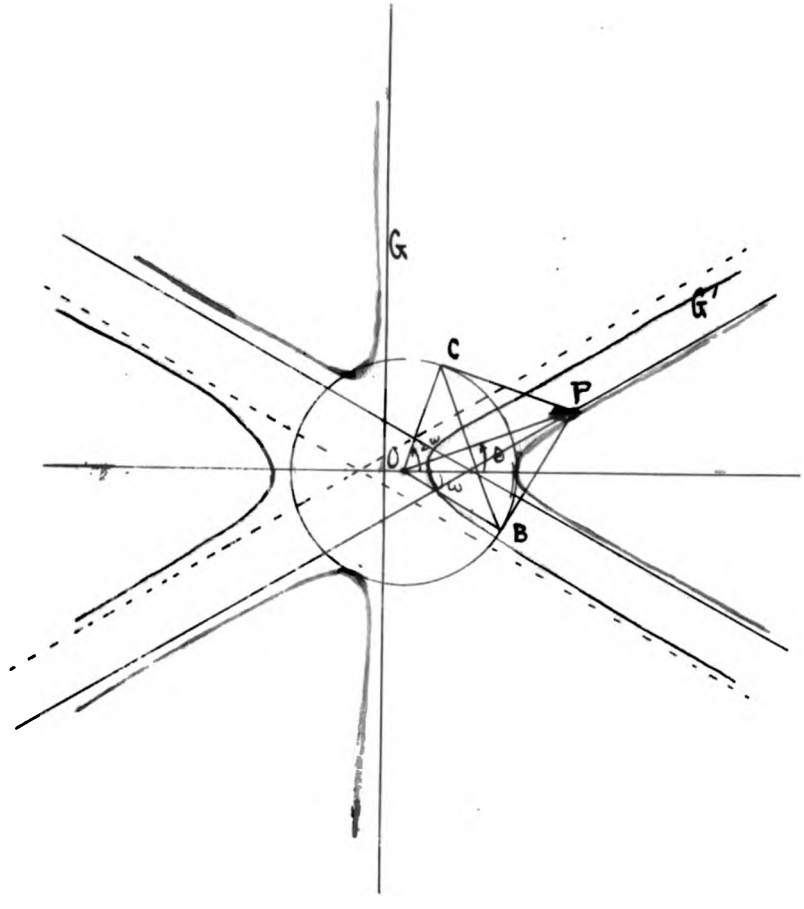

fig. 24.

The tangents to A and C intersect in a point P of the curve. This is the trisectrix of Longchamps. Its polar equation is

(6) $$\rho \cos 3\theta = r,$$

or in cartesian coordinates

(7) $$(x + \tfrac{1}{3}r)(3y^2 - x^2) + \tfrac{4}{3}rx^2 = 0.$$

The line G has the equation $X = -\frac{1}{3}r$; the hyperbola G' the

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equation $y^2 = \frac{4}{9}rx + \frac{1}{3}x$. Besides G the curve has two asymptotes, $(x - \frac{2}{3}r) \pm y\sqrt{3} = 0$, which are parallel to the asymptotes of G' and form with G an equilateral triangle. The curve has three congruent branches, which divide the circle into three arcs of 120 degrees each. O is an isolated point of the curve. The tangents at O are the lines $x^2 + y^2 = 0$; O is a focus of the cubic.

The second case was also constructed by G. de Longchamps and was called the mixed cubic. L. Henkel worked earlier with the same cubic and called it the tangent curve of the parabola. We shall call the curve S the tangent curve of a curve s'. The construction is as follows. Draw a tangent T to any point A of S. The intersection of A and a line parallel to T passing through the origin O is a point of the curve. If S has the equation

(8) $\qquad y = f(x),$

the equation of S' will be

(9) $\qquad y = xf'(x).$

In this way we get the equation of the parabola P,

(10) $\qquad y^2 = 2(x - k)p,$

and the equation of the tangent curve

(11) $\qquad (x - k)y^2 - \frac{1}{2}px^2 = 0.$ $\qquad$ (fig.25)

The third curve does not come in general from a geometric definition, but it has a certain historic interest. Its equation is

(12) $\qquad x^3 + y^3 = 3axy.$

Descarte used it as one of the first examples for his coordinate geometry. The origin is at the double point. We can construct

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the curve by subtracting the vectors from G', the ellipse, to G a straight line. Descartes considered only the loop of the curve and called it the folium, and the curve has been since known as the cartesian folium or the Descartes leaf(fig.26)."

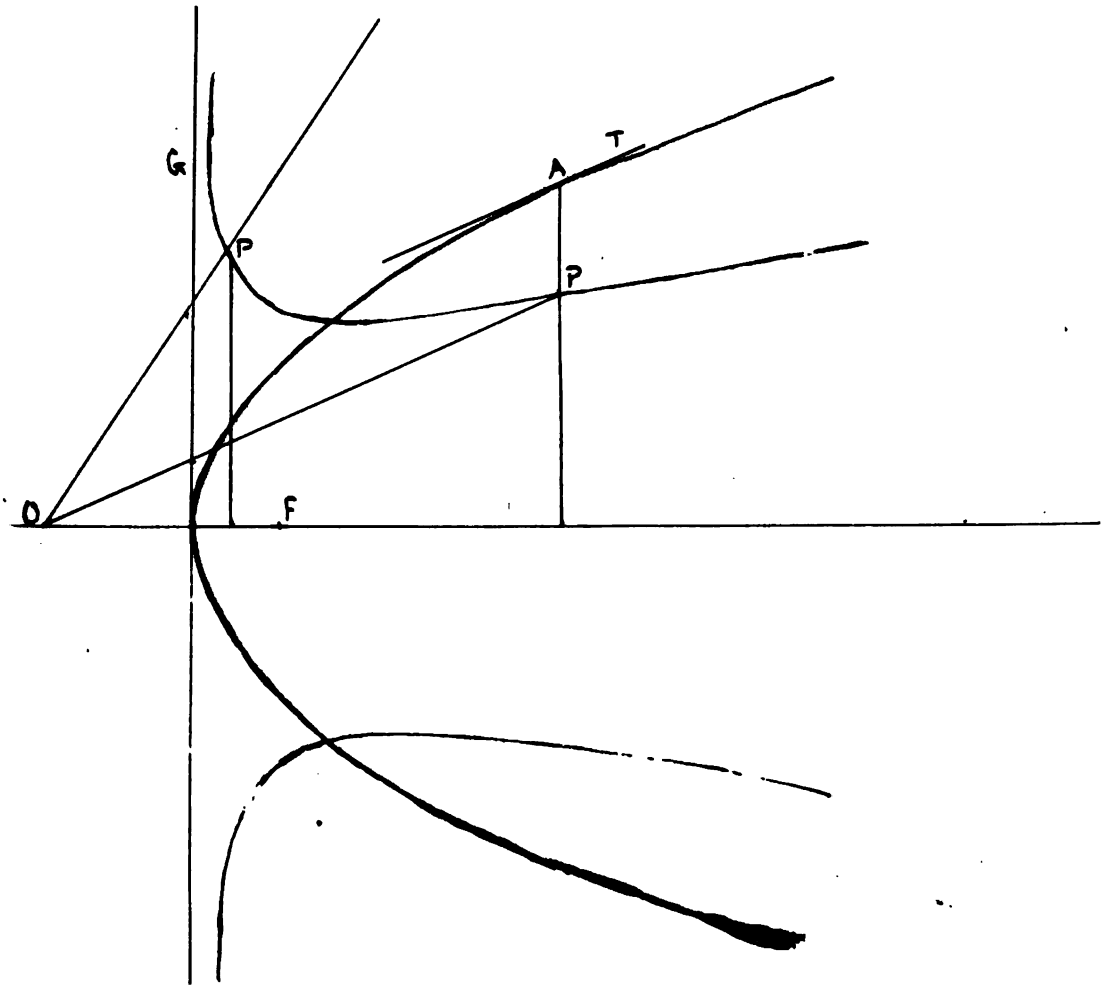

fig.25.

Wieleitner p51.

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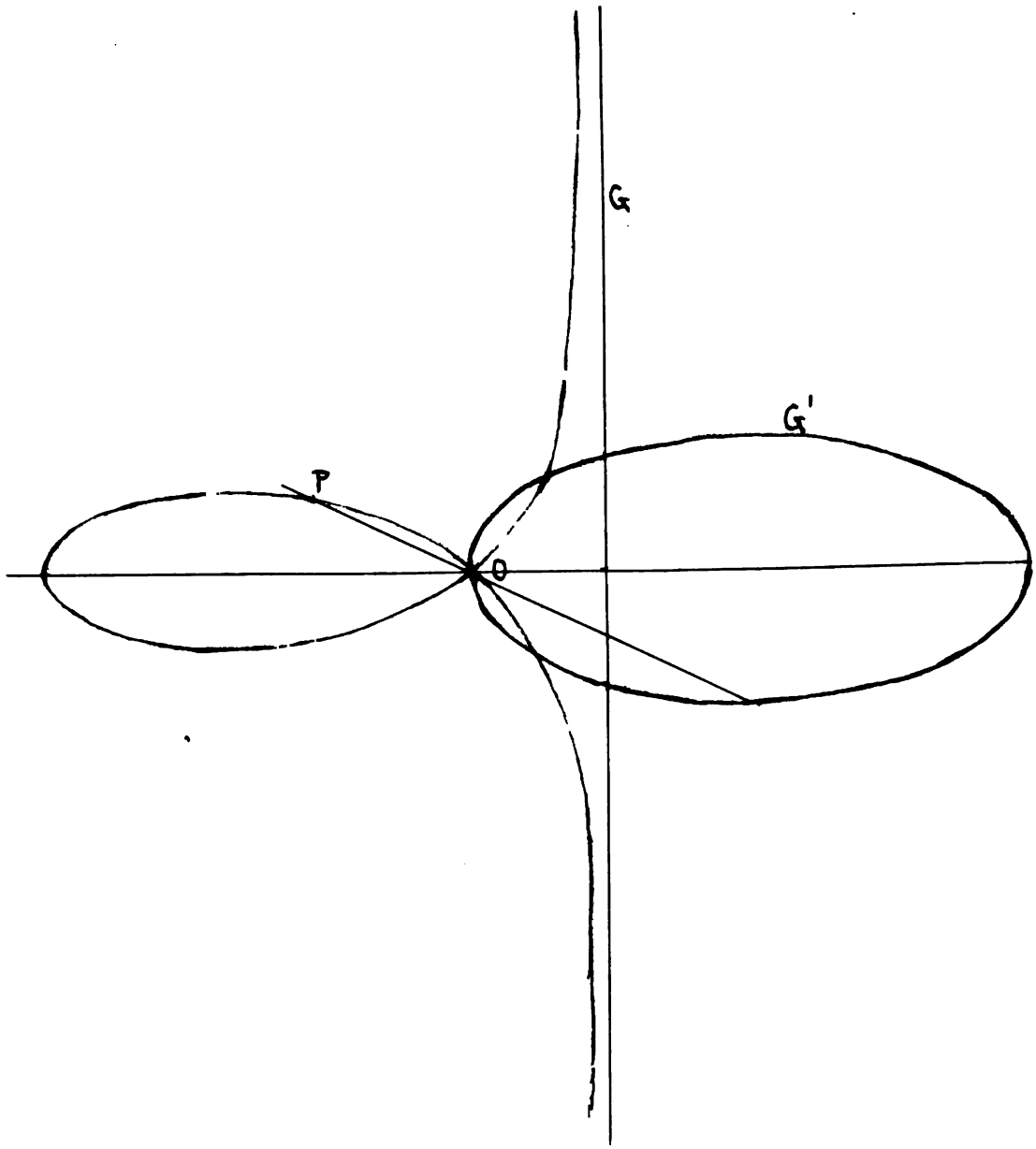

fig.26.

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Two Other Types of Rational Cubics.

If we consider all the curves of the third order with a double point, we must mention mention two families of curves. The first we shall call the normal curve of the parabola. Its form is similar to the tangent curve of L. Henkel. Let the equation equation of the parabola P be

(1) $y^2 = 2(x - k)p.$

Draw to each point A of P the normal N and cut the ordinate of A with a line through the origin O parallel to N. The intersection P describes the curve(fig.27),

(2) $2x^3 = 2kx^2 + py^2.$

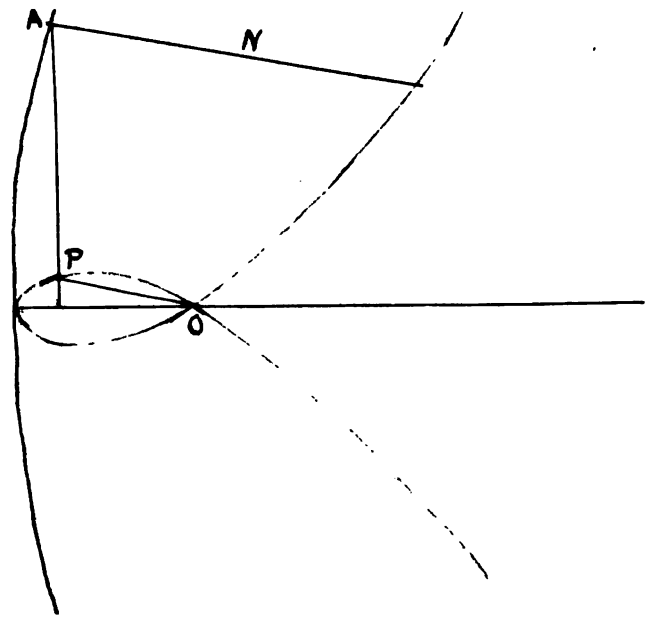

fig.27.

The curve is symmetrical to line OP, and has a double point at O. It belongs to Newtons Divergent parabolas. The double point takes different forms as the position of P changes. If

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$k = 0$, it is a cusp, and the cubic is known as the Neil parabola.

The second type of curve we get from a circle K and a straight line G by means of a transformation. P.H.Shonte investigated this curve and called it MacLaurin's transformation.* Given a curve K, a line G, and three points O, O', O". A line through O cuts K in Q, G in R. Connect Q with O',R with O";O'Q and O"R intersect in P which describes the curve. Now take K a circle, O on K, G, the perpendicular through O. Let $OB = b$, $OA = a$. Let O' be the infinitely distant point from A, O", the infinitely distant point from B, we then obtain the construction in figure 28. Let angle $QOB = e$, and the coordinates of P are

(10) $x = b\cos e$, $y = a\tan e$.

If $a = b$, the curve is known as the Versiera which Agnesi wrote about, and which was named by Guido Grandi in 1718. If $a = \frac{1}{2}b$, the curve was called the geometrical quadratrix by Ozanum.

*Wieleitner p.58.

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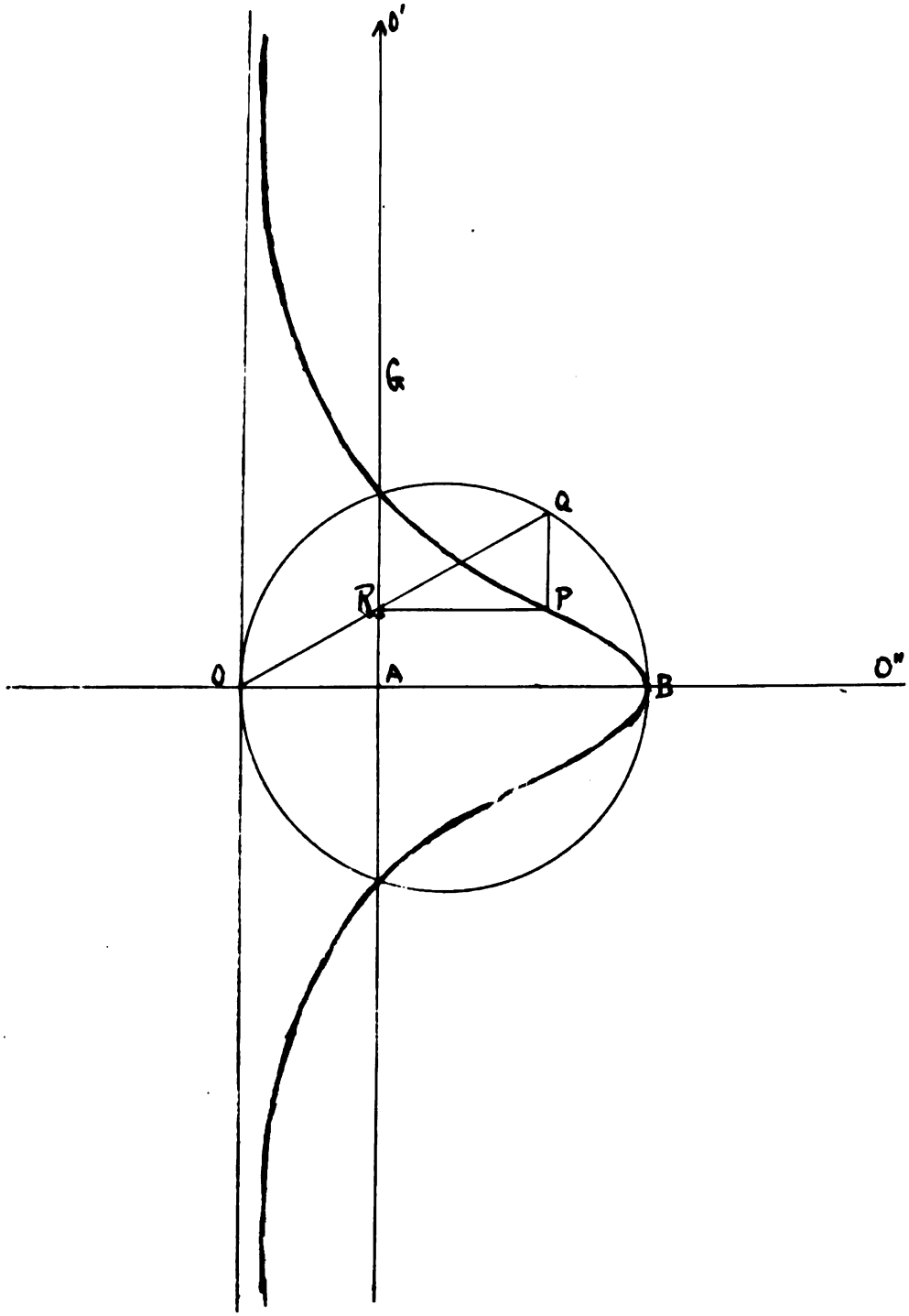

fig.28.

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Chapter 4.

Special Cases of the Concheid.

In chapter two we considered the concheid as a general case. Just as in chapter three we considered the special cases of the cisseid, so in chapter four we shall consider the special cases of the concheid. Let G be a given curve, O afixed point, to find the concheid of G. We will take as the radius vector OP of G a line RP = l moving with the constant angle OPR = w. (fig. 29) To find the instantaeous center M, we will draw the normal to G at the point P and the perpendicular to OP at O, the intersection will give M. M is independent of l and w.* We can then say the normal of every concheid of a base curve G goes through the end of the polar subnormal of G. We will consider special cases of this construction, first taking G as astraight line(fig. 29). To construct the center of curvature C for a point R which lies on OP at a distance PR = l from P, draw PM perpendicular to G, OM perpendicular to OP. M is the instantaneous center, RM the normal, MB is perpendicular to PM, PB - BA, N is the middle point of MA, MB' is perpendicular to MR, B'N cuts the normal MR in C. N remains constant for all values of l.

We will now find the equation of the oblique concheid described by the point R, when Gis a straight line. We know it will be a quartic. If the distance of the pole O from G is OO' = a, and if we place the variable angle OPO' = e, the coordinates of R, considering O as the origin, are

*Wieleitner p .64.

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(1) $x = a\,\mathrm{ctg}\,e - l\cos(w+e)$

$y = -l\sin(w+e).$

Eliminating e we get

(2) $(xy\cos w-(y^2+ay-l^2)\sin w)^2=(x\sin w+y\cos w+a\cos w)^2(l^2-y^2)$

or

$H^2 = A^2(l^2 - y^2).$

We have a quartic with two double points, D and D', where the line A which goes through O cuts the hyperbola H(fig. 31).

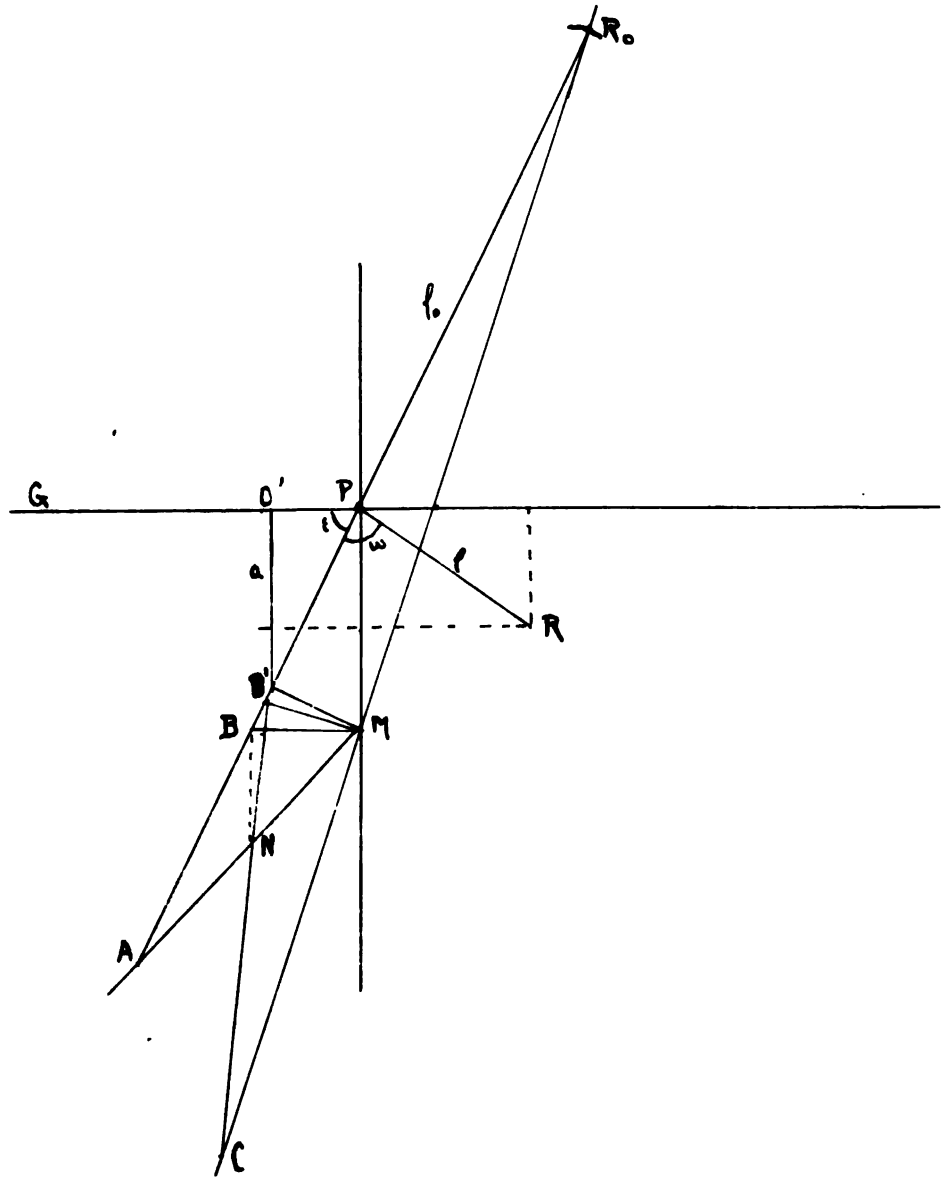

Fig. 29.

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In addition there is a node at infinity in the direction of G with the asymptotes $y = \pm l \sin w$. The curve goes through the circular points. It is therefore rational and circular. The lines $y = \pm l$ are horizontal tangents. Between these lies the curves. The double point D is always a node; D' is either a node, cusp, or isolated point as the relative distance from a, w, and l. The line PR envelopes a parabola with focus at O. O is also the extraordinary focus of the conchoid, that is the asymptotes go through it at the infinite circular points. If we place $x = \pm i(y + a)$ in (2), the coefficients of y and y vanish and the equation has two roots $z = 0$.

If we let $w = 0$, we have another special case. This gives the conchoid of Nicomedes whose equation is

(3) $$x^2y^2 = (y + a)^2(l^2 - y^2).$$

This is the eldest conchoid. Its equation in polar coordinates is

(3*) $$\rho = a/\sin\theta + l.$$

The two parallel asymptotes of the conchoid form the x axis. The curve is symetrical to the y axis, D' moves to O. D is on the x axis at infinity. At O there is a node, cusp, or isolated point as $l > , = , < a$. The form with the node is shown in figure 30. One branch lies above the x axis, another below, both are mussel like in form.

Another special case is obtained if we let $w = \frac{1}{2}\pi$.

(4) $$(xy - y^2 - ay + l^2) = (x + y + a)^2(l^2 - y^2).$$

This curve(fig.31) was discovered by A. Dürer, and was called*

*Wieleitner p 65.

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the "Muschellinie". A.Dürer was born in Nürnberg in 1525. The double point D' is a node, cusp, or isoslated point as 1>,-,<2a.

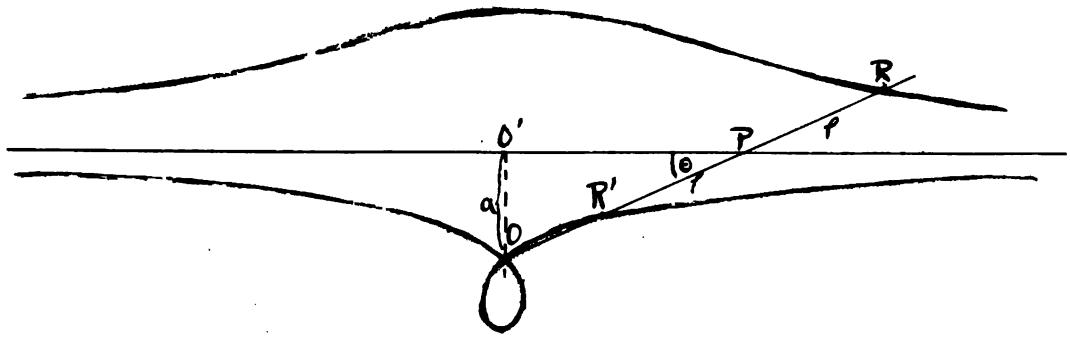

fig.30.

If we let $w = \frac{1}{3}\pi$ in the equation (2), we get the curve of figure 32 with the equation

(5) $(y^2 + ay - l^2) = x^2(l^2 - y^2)$.

This curve was invented by J.Neuberg and was called the ortho concheid of the straight line. Like the concheid of Nichomedes, it is symmetrical in form. It has two finite double points, the one at D always a node, D' always an isolated point located on the y axis with the ordinate $y = -\frac{1}{2}(a \pm \sqrt{a^2 + 4l^2})$. From this we can see that the isolated point can never fall at the pole O.

Using the x axis as the fundamantal curve, we can get the pole curve of concheidal motion. The fixed path of the pole is the position of momentary centers M in the fixed plane U. For a given position of the moving line OP the coordinates

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are

(6) $x = a\,\mathrm{ctg}\,e,\ y = -a/\sin^2 e.$

Eliminating e

(6*) $x^2 + a(y + a) = 0.$

This is a parabola with O as vertex and OO' as axis.

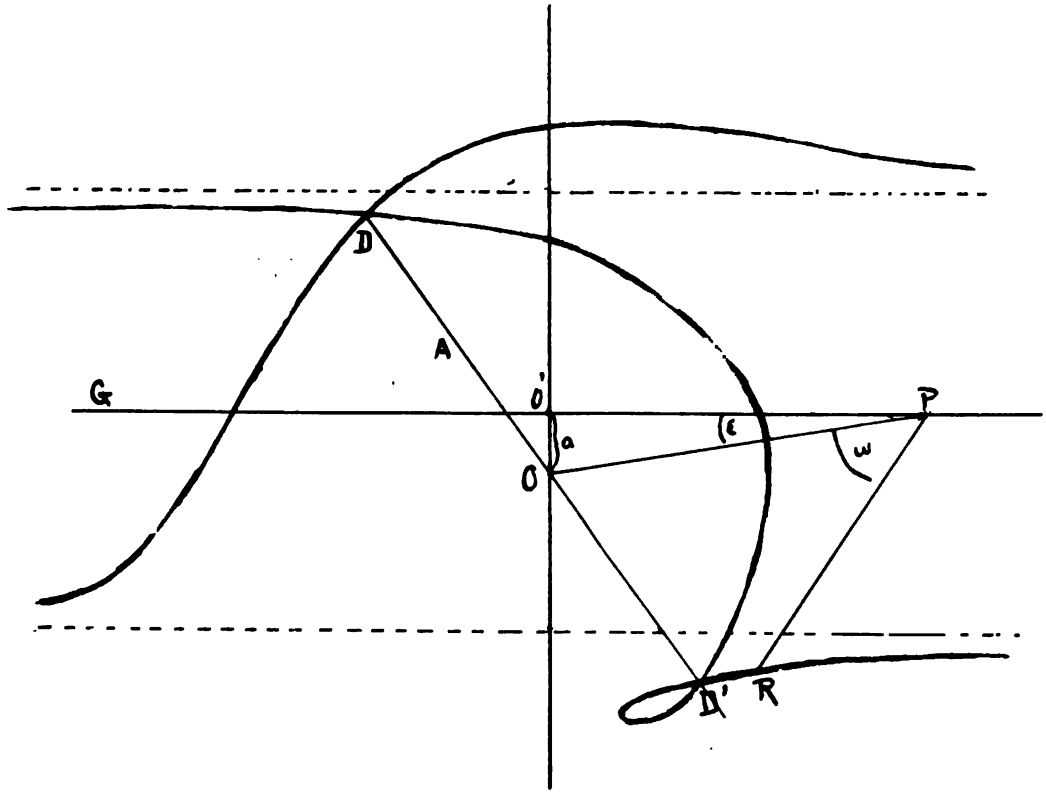

fig.31.

The moving path of the pole is the inverse of the cencheidal construction. Given a line OH and a fixed point P on the line. A right angle with vertex at O moves so that one leg always goes through P, while a fixed point O of the other moves along OH. The momentary centary M is constructed in this way. If we take OP as the n axis, and a line perpendicular at

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P as the s axis, the coordinates of M are

(7) $s = a\cos e/\sin^2 e, \quad n = -a/\sin e.$

Eliminating e we get the equation of the moving polar path,

(7*) $a^2(s^2 + n^2) = n^4.$

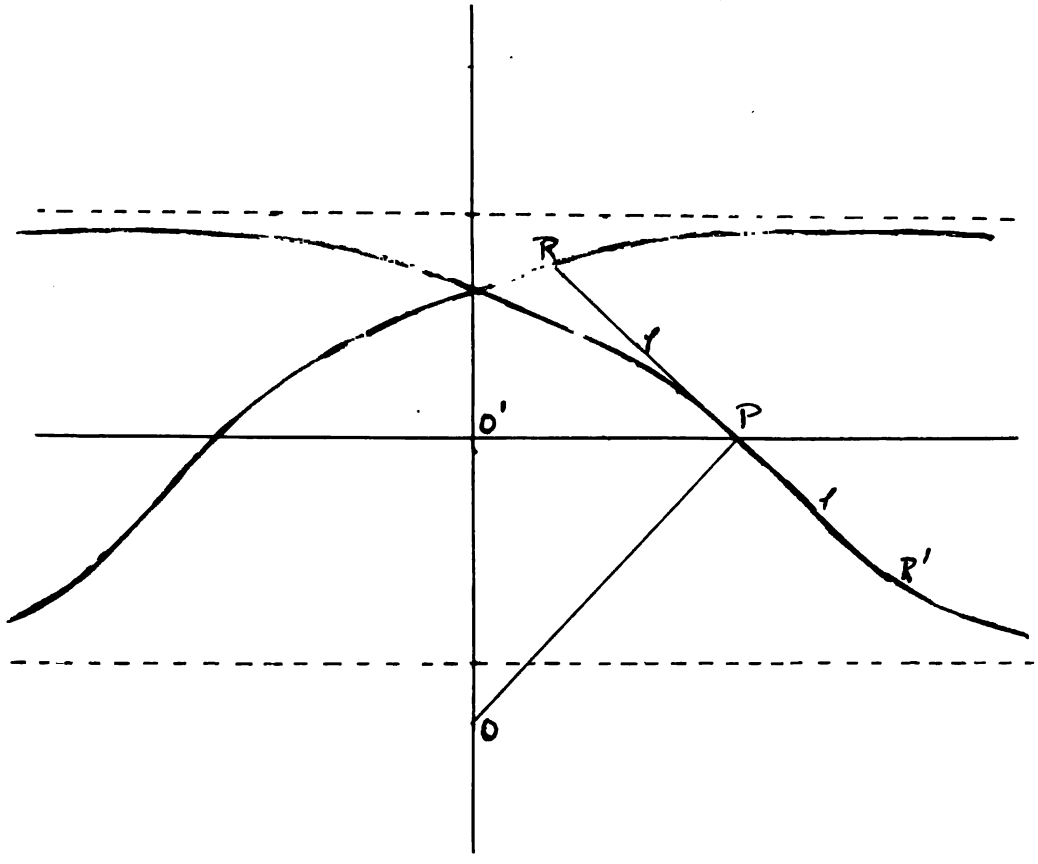

fig.32.

This quartic (fig.33) where P falls on O' and the respective axes come together has a parabola (6*), and all the points of the plane U' in which (7*) lies describes a concheid of the line O'X. All points of the n axis describe Nicomedes' concheid. The quartic (7*) has an isolated point at O' with the isotropic lines as tangents; at the infinite point the s axis makes a tac-node with the infinitely distant

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line as tangent. Its polar equation is

$$\rho = a/\sin^2\theta.$$

according to P. tannery this curve is the curve which Eudoxus of Knidos called the "Kampyla".This was used as well as the concheid of Nicomedes to solve the Delian problem of the duplication of the cube.

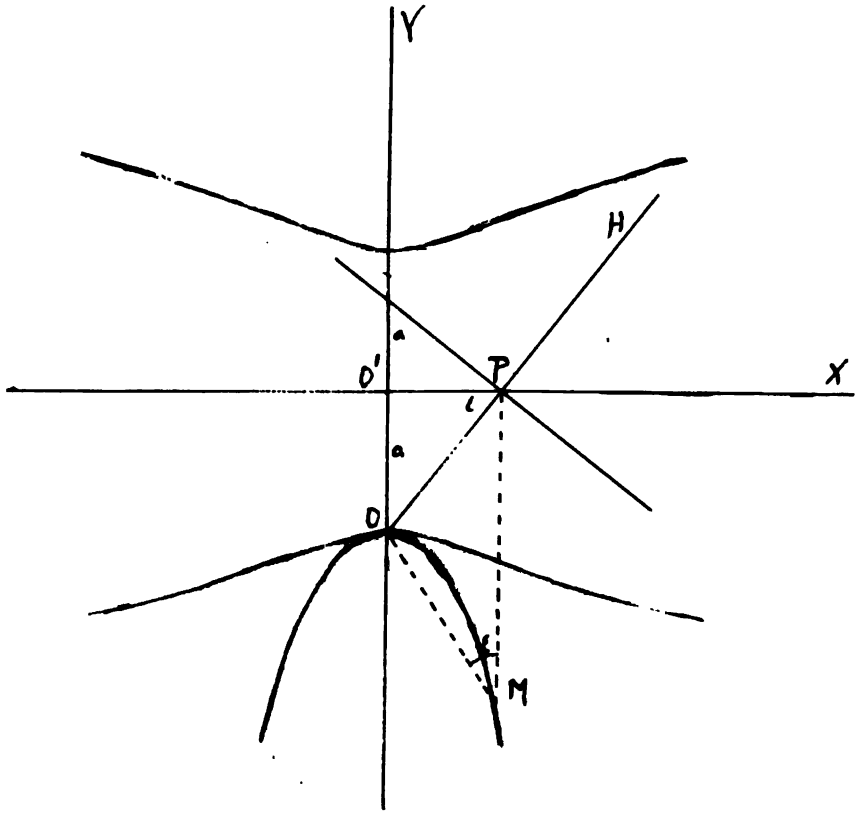

fig.33.

We will find the trajectory of the point in U when the parabola rolls on the Kampyla. The inverse of the Kampyla refered to its middle point is the well known curve whose equation is

(8) $$\rho = a\sin^2\theta.$$

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or

(8*) $(x^2 + y^2)^3 = a^2y^4$.

F. Münger named it the double egg curve(fig.34). This curve has a fourfold point at the origin with a fourfold tangent. Its polar equation is $\rho = \frac{1}{2}a - \frac{1}{2}a\cos 2\theta$.

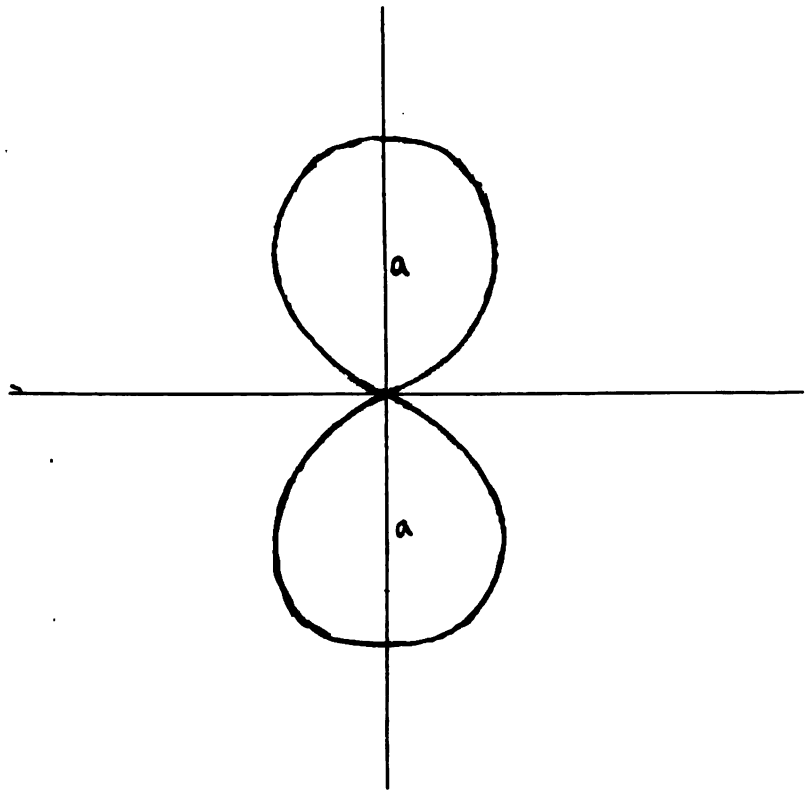

fig.34.

Let G'be a fixed line, O a fixed point of the plane U. In the moving plane U' let G be a fixed line and P a fixed point. (fig.35) Draw U' over U so that G always goes through O while P remains on G'. To find the equation of the trajectory of a desired point Q of the plane U', we will let QP = l and the angle that QP makes with the perpendicular PP'(=b) of G is W. With O' as an origin, the coordinates of Q are

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(1) $x = a\,\mathrm{ctg}\,e + b/\sin e + l\sin(W-e)$

$y = -l\cos(W+e)$.

Eliminating e, we get the equation of the trajectory,

(2) $(xy\sin W + (y^2+ay-l^2)\cos W-bl)^2 = (x\cos W-y\sin W-a\sin W)^2(l^2-y^2)$.

This is a rational quartic, which is not very different from the curve (2) described before. If $b = 0$ and $W = w-\frac{1}{2}\pi$, it is similar.

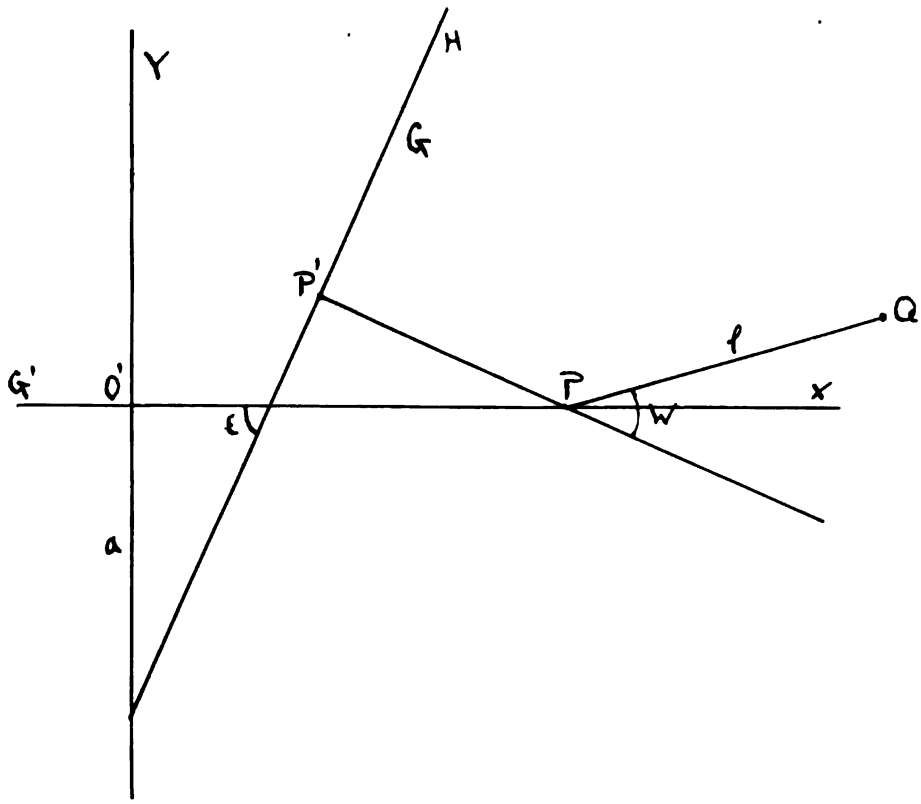

fig.35.

There are three important cases of this motion, (1) when $b = 0$ which is the conchoidal motion described before, (2) the case $a = 0$, the inverse of the conchoidal motion, (3) the case $a = b$, the so called symmetrical sliding motion. We will first consider the inverse of the conchoidal motion. From the general equation we get for the trajectory of a given point

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the curve

(3) $$(xy\sin W+(y^2-l^2)\cos W-bl)^2 = (x\cos W-y\sin W)^2(l^2-y^2).$$

The motion is really the rolling of a parbola on a Kampyla. The curve (3) is symmetric with respect to O; the equation is not changed if x and y are both changed in sign. It is symetrical to the axesonly if the xy factor vanishes. To accomplish this sinW must equal O.PQ falls on PP'. The curve has the equation

(4) $$(y^2 - l(b+l))^2 = x^2(l^2-y^2).$$

If l = b, the point P' describes the curve

(5) $$y^2(x^2+ y^2) = b^2x^2;$$

or

(5*) $$\rho = b\,\mathrm{ctg}\,\theta.$$

This curve was discovered by G.van Gutscheven, a teacher of Descartes, and by J.Barrow(1670). J.Aubrey called it the Kappa curve.* The curve can also be defined as the locus of the point of contact P' of the tangent of O on the circle of radius b, whose center P moves on the x axis. The Kappa curve has parallel asymptotes y = ±b, and at the origin a tac-node with the y axis as tangent. It is a rational circular quartic. (fig.36)

We get the tac-node at O in equation (3) when the constant disappears or when lcosW = -b. That is always the case when Q lies on OP'. The trajecteries are the usual concheids of the Kappa curve which are called oblique Kappa curves. If we let lsinW = l', equation (3) becomes

(6) $$y^2(l'x - by)^2 = (bx - l'y)(b^2 - l'^2-y^2),$$

or

*Wieleitner p.74.

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(6*) $\rho = b\,\mathrm{cotg}\,\theta + l'$.

From these equations we can see that their are two oblique Kappa curves; fer (6) remains unchanged if we turn θ through $\theta -$, and (6) changes if l' changes sign. The oblique Kappa curve is generated by moving a fixed angle OQP in such a way that a right angle OPP moves along the Kappa curve. Q describes the curve.

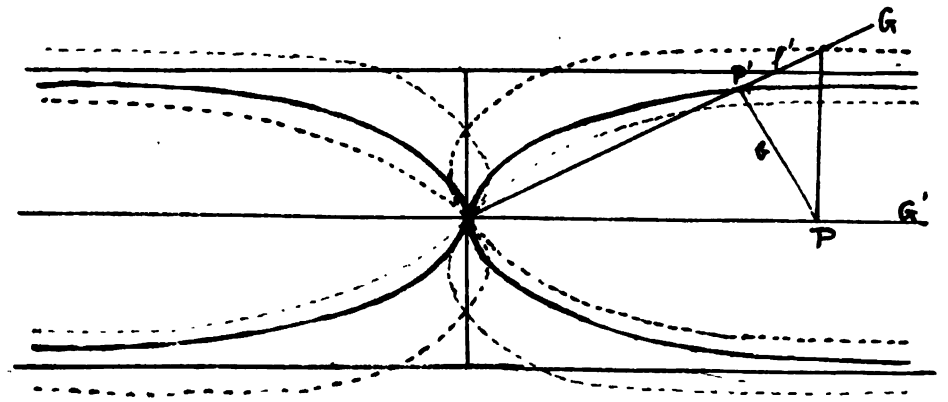

fig.36.

The Conchoid of the Circle.

The motion described previously is called sliding crank motion if instead of the fixed line G of the plane U, we take a fixed circle on which the fixed point of the plane U' moves, the line G slides through O(fig.37). The trajectories of the points of U' are curves of the sixth order. We are especially interested in the case when P lies on G(PP' = O). The point P moves on the circle K, while G goes through O. The path of the point of U' refered to the pole O is known as hte conchoid of the circle K. Allpoints of G describe the or-

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dinary conchoid of the circle. We can specialize further and consider the case when O lies on the circle K(fig.38). The curve generated is defined as follows: on a circle K lies a point O. On each radius vector OP moves a constant length PQ-l making a constant angle W with OP. Q describes the oblique

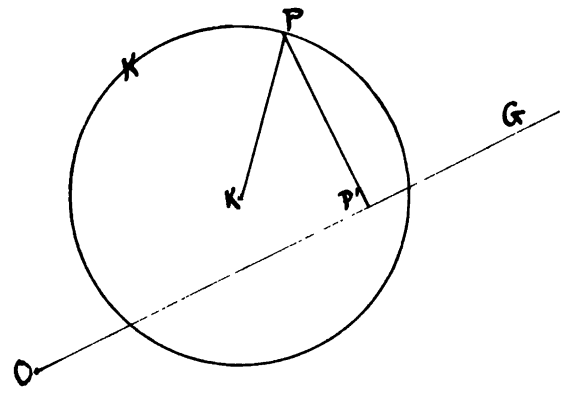

Fig.37.

conchoid. Extend PQ so that it cuts the circle in O'. For each position of the radius vector OP, O' is a fixed point on K.

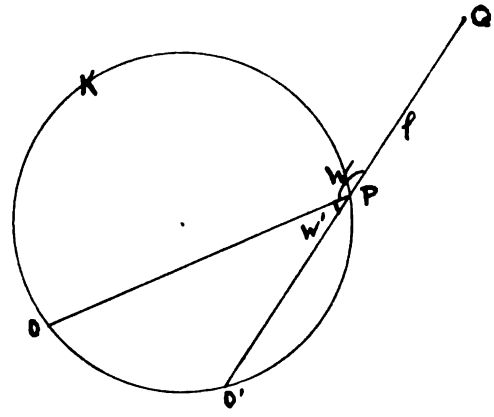

fig.38.

Then angle OPO' = W' = π-W is constant when W is constant. Each oblique conchoid of a circle refered to a point of the circle as a pole is an ordinary conchoid of the same circle

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refered to another point on the circumpherence. If we take OK (=r) as the polar axis, the equation of the locus of Q is

(1) $\rho = 2r\cos\theta + l,$

or

(1*) $(x^2 + y^2 - 2rx)^2 = l^2(x^2 + y^2).$

To get the mementary center M draw a perpendicular to G at O, a perpendicular to K at P, and M is the intersection of the perpendiculars. Hence the fixed pole is the circle about P with the radius 2r (fig.39). The curve (1) which is called Pascal's roll after its discoverer, Stephan Pascal, the father of Blaise Pascal, can therefore be generated if we let a circle

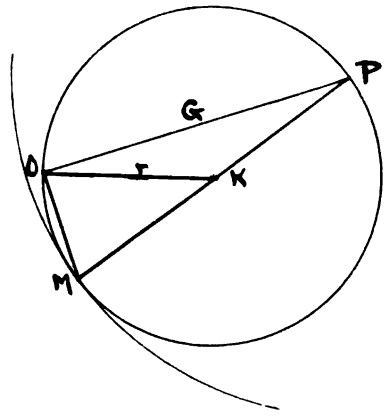

fig.39.

roll on another of half its radius, so that they are always in contact. These curves belong to the great family of cycloids.

To investigate the form of the Pascal rolls, we bring equation (1*) to the form

(1#) $(x^2 + y^2)^2 - 4rx(x^2 + y^2) - ((l^2 - 4r^2)x^2 + l^2y^2) = 0.$

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The origin is a node for $l<2r$, a cusp for $l=2r$, an isolated point for $l>2r$. To construct the curve, use O as a pole, let P move about the circle and measure off on either side of OP Q and Q' at the distance l. Q and Q' generate the curve. In figure 40, $l<2r$; in figure 41 $l = 2r$; in figure 42 $l>2r$. In the case of the cusp the curve is called the cardiode. According

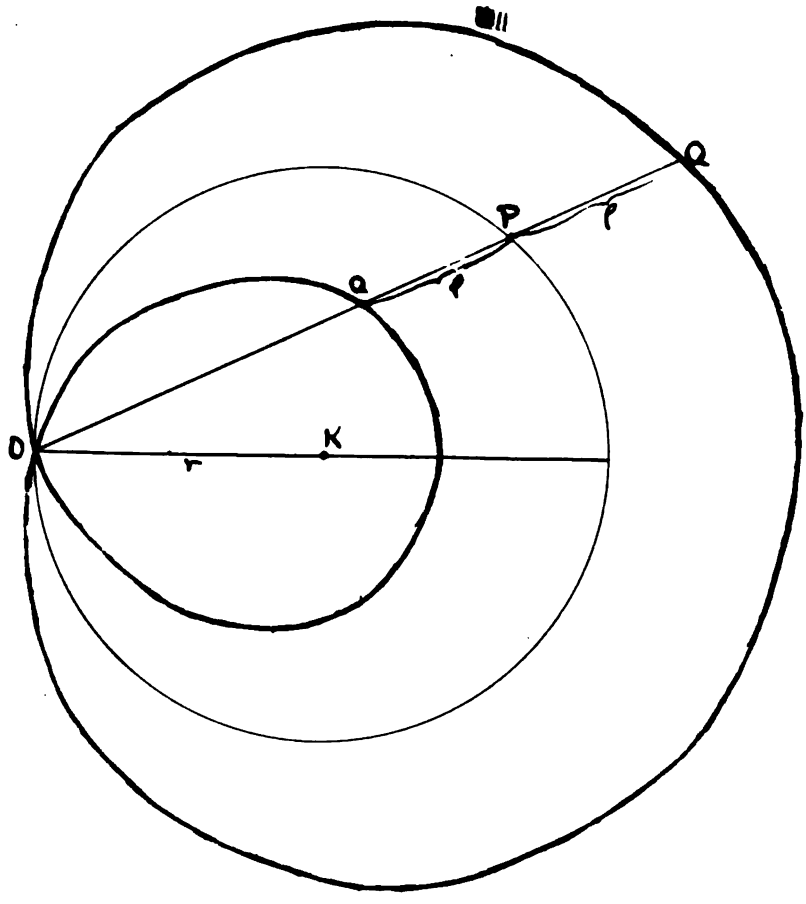

fig.40.

to the Plücker formulae the Pascal roll is one of fourth class, (the cardiode of third) and has two points of inflection, (the cardiode none) and a double tangent.*

*Wieleitner p.89.

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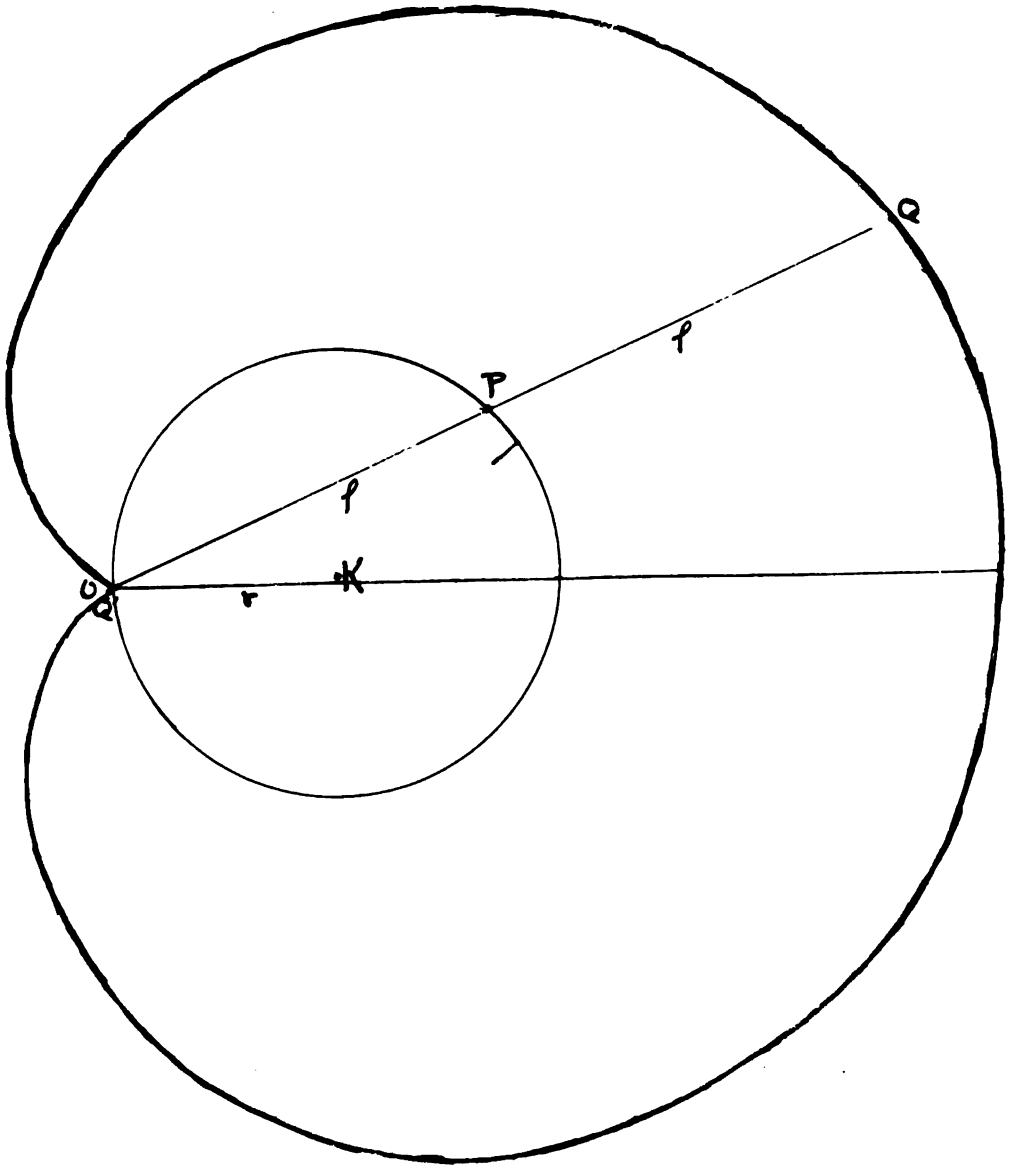

fig.41.

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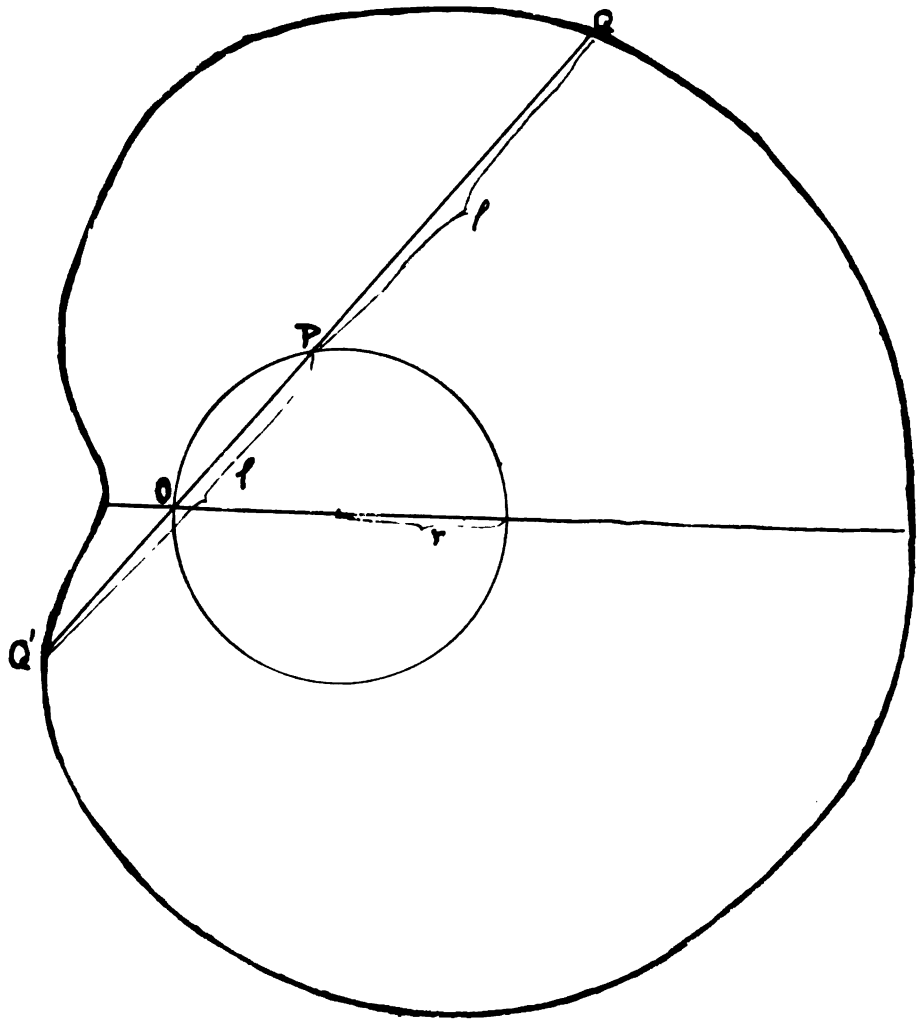

fig.42.

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Concheids of Conic Sections.

We will consider only two special cases of the concheid of the conic section. First we will let the pole be at the vertex of the conic section. This has the equation

(1) $y^2 = 2px - (e^2 - 1)x$,

where e is the eccentricity. For the ellipse $p = -b^2/a$. In polar coordinates the equation becomes

(1*) $\rho = 2p\cos\theta/(\sin^2\theta - q\cos^2\theta)$,

where $e^2 - 1 = q$.

Thus the equation of the vertex concheid of this conic section becomes

(2) $\rho = 2p\cos\theta/(\sin^2\theta - q\cos^2\theta) + l$,

or

(2*) $(x^2 + y^2)(y - qx^2 - 2px)^2 = l^2(y^2 - qx^2)$.

There is a fourfold point at the pole. If the foundation curve is an ellipse and the pole lies at the vertex of the major axis, the concheid, if $l<2a$, appears somewhat similar to the Pascal roll with nodes. In that case a circle about the pole with radius l cuts the ellipse in two real points only, two of the tangents at the fourfold point are also imaginary. If $q = -1$, we can divide out the factor, $x^2 + y^2$, and we get the equation of the Pascal form. If we take the pole on the minor axis, the circle about the pole with l as radius cuts the ellipse in four real points(fig.43.). Figure 44 shows the curve for the hyperbola. In the case of the parbola only two branches of the double point are real.*

*Wieleitner p.106.

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In the second place we will consider the concheid of a focus. The polar equation of a conic section at a focus is

(3) $\rho = p/(1+e\cos\theta)$,

and the polar equation of the concheid is

(4) $\rho = p/(1+e\cos\theta) + 1$,

and the cartesian equation is

(4*) $(x^2+y^2-ex)^2-(x^2+y^2)(p+1-ex)^2=0$.

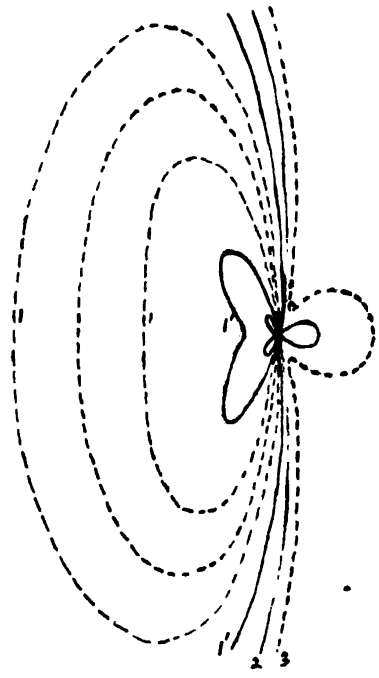

Fig.43.

There will be a double point at the pole. If we draw a circle with radius 1 about the pole, we know that the double point, when the ellipse is the base curve, is a node, if $a+e>1>a-e$, where $e=\sqrt{a^2-b^2}$. Equation (4*) shows that the two points where

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the circle, $x^2 + y^2 - elx = 0$, cuts the line, $p + l - ex = 0$, are double points of the curve. Figure 45 pictures the focus concheid of an ellipse.

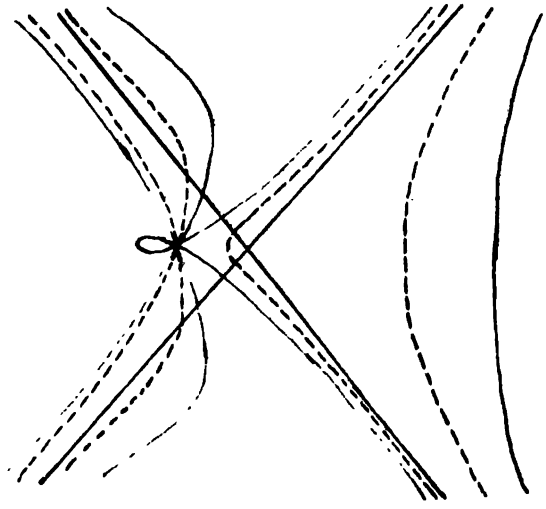

fig.44.

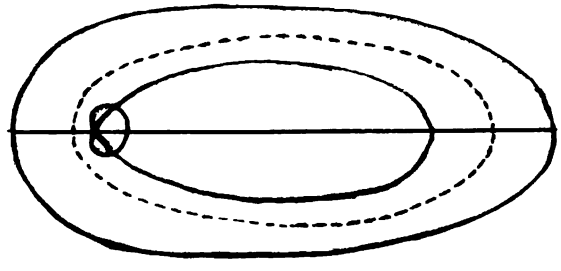

fig.45.

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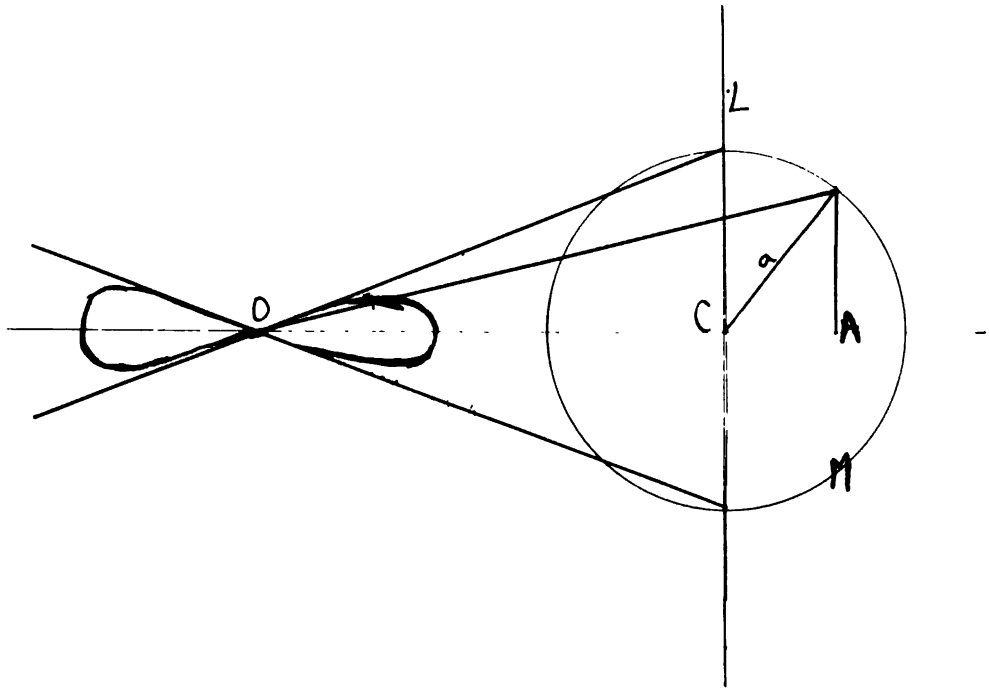
Chapter 5.

The Curves in Three Dimensions.

In the previous chapter we have considered space of two dimensions, in this chapter we shall pass to three dimensions and consider briefly cissoidal surfaces and cissoidal space curves. Let M be a circle of radius a, L a straight line passing through the center of the circle. The pole O is on a line perpendicular to the line l and at a distance h from C, the center of the circle. We can obtain the cissoid by subtracting the radii vectors from O to M, and from O to L. The



radius vector r from Oto L is hcosθ where h is the distance from O to L. The radius vector r'(from O to M) can be found as follows.

$$MA = r'\sin\theta.$$

$$MC = a.$$

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$$CA = r'\cos\theta - h.$$

$$\overline{MC}^2 = \overline{MA}^2 + \overline{CA}^2.$$

Therefore $r'^2\sin^2\theta + h^2 + r'^2\cos^2\theta + 2r'h\cos\theta = a^2.$

$$r'^2 + h^2 - 2r'h\cos\theta = a^2.$$

Subtracting r' from r, we get the equation of the cissoid,

$$R = \sqrt{a^2 - h^2\sin^2\theta},$$

or $$(x^2 + y^2)^2 = a^2(x^2 + y^2) - h^2y^2.$$

Rotate the cissoid about the x axis to get the cissoidal surface, and the equation becomes

$$(x^2 + y^2 + z^2)^2 = a(x^2 + y^2 + z^2) - h^2(y^2 + z^2).$$

If we mave the line parallel to itself to a point between D and C, the cissoid changes shape(fig.47) and has the equation,

$$R = h'\cos\theta - h\cos\theta \pm\sqrt{a^2 - h^2\sin^2\theta},$$

where h' is the distance from O to L. $h > h' > h-a$.

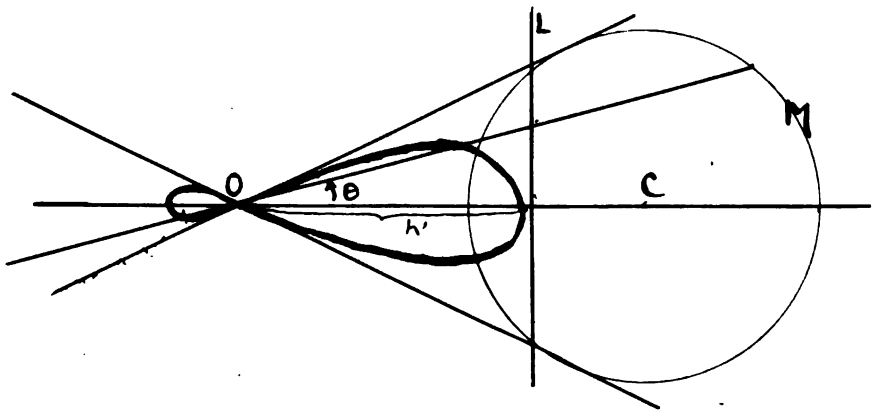

fig.47.

In rectangular coordinates the equation is

$$(x^2 + y^2 - h'y + hy) = \pm a(x^2+y^2) - h^2x^2.$$

Revolving about the x axis, we get the equation of the surface,

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$$(x^2 + y^2 + z^2 + h'\sqrt{y + z} + h\sqrt{y + z})^2 = a^2x^2 + a^2y^2 + a^2z^2 - h^2x^2.$$

Move L, keeping it perpendicular to OC until it is tangent to M at D, the cissoid now has but one branch. Fig.48.

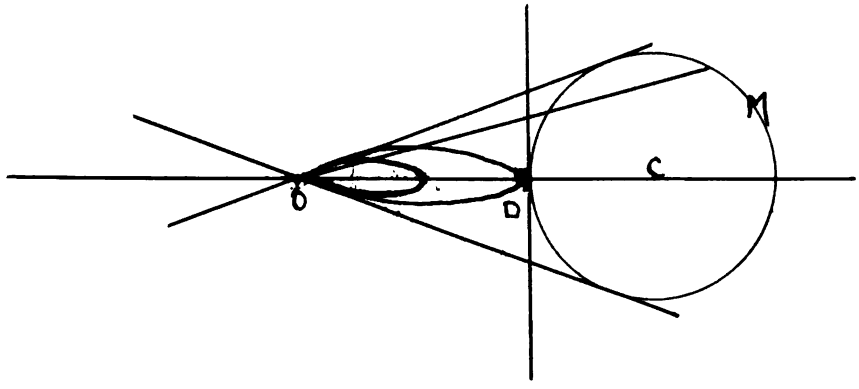

fig.48.

Its equation is

$$R = -a\cos\theta \pm \sqrt{a^2 - h^2\sin^2\theta}$$

or

$$(x^2 + y^2 + z^2 + a\sqrt{y^2 - z^2}) = a^2 - h^2x^2.$$

In the same way we can move the line L in the opposite direction; the form of the resulting curves are shown in figures 49 and 50.

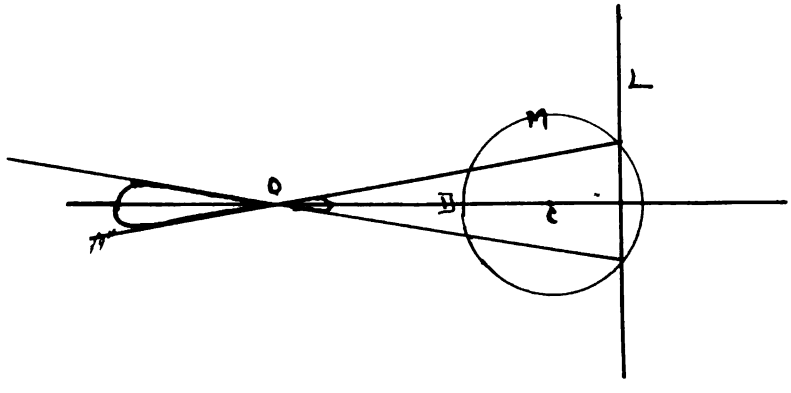

fig.49.

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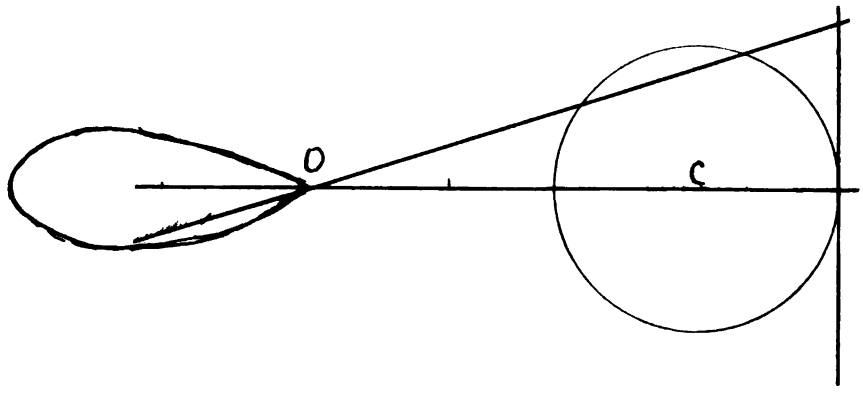

fig.50.

We shall now consider the form of the cissoidal surface which we shall call pedal surfaces. Given a circle C and a straight line L tangent to C and a pole O. Drop a perpendicular from O to the tangent, intersecting the tangent at P. Let L move about C and the point P describes a curve.

Case 1. Let $OC > a$ where a is radius of the circle.

$\Delta OPX \sim \Delta CAX$,

therefore $\angle\theta = \angle\theta$.

$\angle OPX = 90^\circ$ by construction.

$\angle CAX = 90^\circ$. Radius perpendicular to a tangent of C.

Therefore $r/a = OX/CX = (h\sin\theta + a)/a$

$$r = h\sin\theta + a:$$

or

$$(x^2 + y - hx)^2 = a(x^2 + y^2).$$

Rotate the curve about the x axis

$$(x^2 + y^2 + z^2 - hx)^2 = a(x^2 + y^2 + z^2)$$

See figures 51 and 52.

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Case 2. Let OC = a, that is, O is on the circle, and we get the curve known as the cardiode. (fig.53)

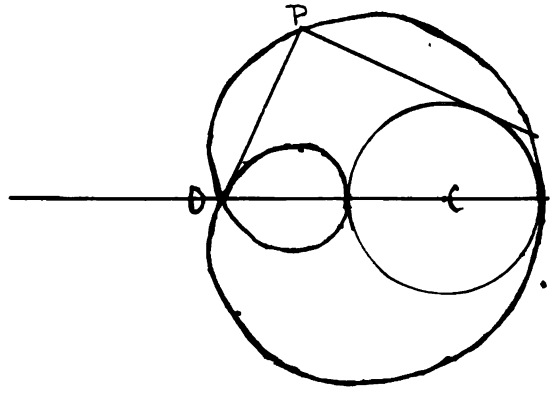

fig.51.

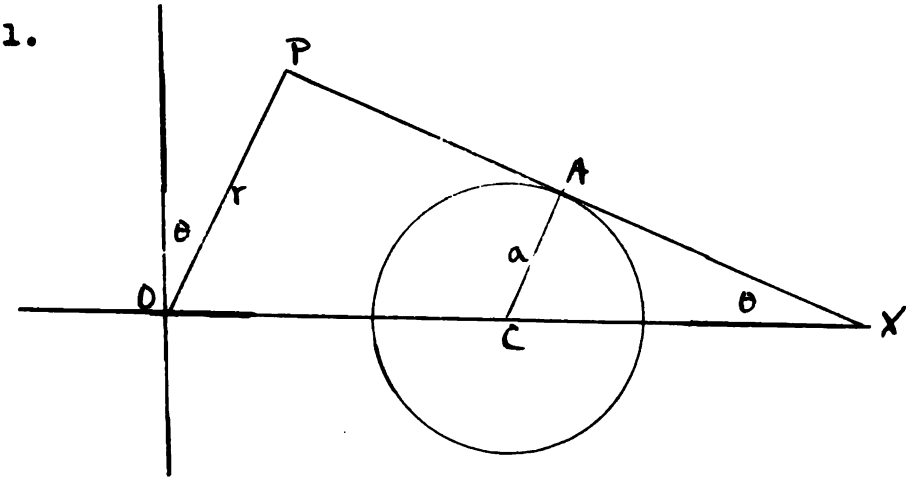

fig.52.

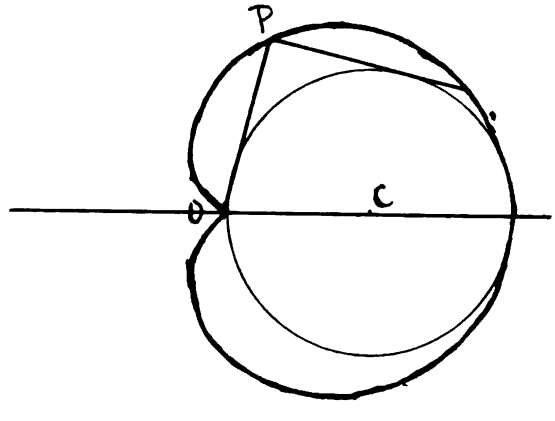

fig.53.

Its equation is

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$(x^2 + y^2 - ax)^2 = a^2(x^2 + y^2)$.

The equation of the surface is

$$(x^2 + y^2 + z^2 - ax)^2 = a^2(x^2 + y^2 + z^2).$$

Case 3. Now let $OC < a$. fig.54. The equations are as in case 1 but here $h < a$.

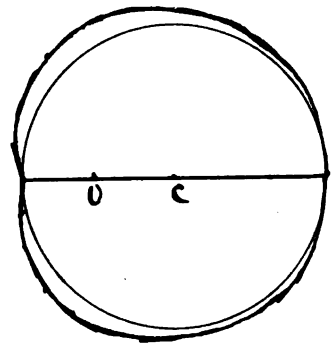

fig.54.

Case4 Let $OC = 0$, that is, the pole coincides with C and the curve generated will coincide with the circle. fig.55.

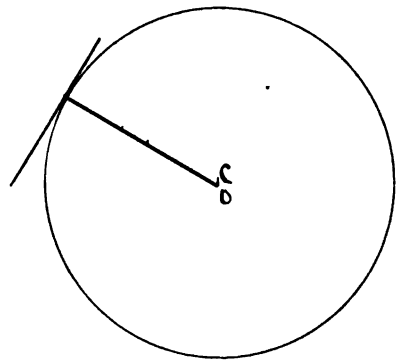

fig.55.

We will find the conchoid of the ellipse in the same manner as we found the conchoid of the circle. The equation of the vertex conchoid of this conic section is from chapter4,

$$(x^2 + y^2)(y^2 - qx^2 - 2px)^2 = l^2(y^2 - qx^2)^2.$$

Revolve this curve about the x axis and the equation of the resulting surface is

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$$(x^2 + y^2 + z^2)(y^2 + z^2 - qx^2 - 2px)^2 = l^2(y^2 + z^2 - qx^2)^2.$$

We shall conclude by a brief consideration of the cissoidal space curves. Take the curve $x = l^3$, $y = l^2$, $z = l$ on the surface of a paraboloid. To generate a new curve, allow the line OP' passing through the origin to move along the curve. The points P' and P'' on this line at a distance a' on either side of the point of intersection P of the curve and the moving line is

$$x/l^3 = y/l^2 = z/l.$$

The curve generated by the points P' and P'' lie on the cone $y^2 = xz$, since the equation of the line OP' satisfies the equation of the cone. To get the equation of the generated curve, consider the sphere $(x - l^3)^2 + (y - l^2)^2 + (z - l)^2 = a^2$. The equations of the desired curve are

$$(x - x^3/y^3)^2 + (y - x^2/y^2)^2 + (z - x/y)^2 = a^2$$

$$y^2 = xz.$$

We will next consider the screw curve on the cylinder, whose equation is

$$x = a\cos u$$

$$y = a\sin u$$

$$z = au\operatorname{ctg}\delta.$$

$$PP' = A'P'\operatorname{ctg}\delta.$$

$$PP' = z.$$

$$A'P' = au. \quad \text{fig.56.}$$

If we generate a curve by allowing a line LL' to move on the surface of the cylinder perpendicular to the curve, the point

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L' at a distance a from the curve and the point L at a distance b generate a curve. The z coordinate remains the same, the x and y are changed and the equation of the resulting screw curve is

$$x = b\cos u$$

$$y = b\sin u$$

$$z = au\,\mathrm{ctg}\,d.$$

If we move the line parallel to the z axis the point L at a distance b generates a third srew curve whose equation is

$$x = a\cos u$$

$$y = a\sin u$$

$$z = (a \pm b)u\,\mathrm{ctg}\,d.$$

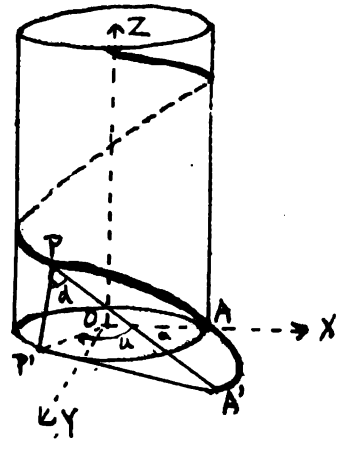

fig.56.

If we move the line LL' at an angle k to the cylinder, the points L'and L generate a pair of screw curves. The new coordinates as seen in figure 57 are

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$$z' = z + r$$

$$x' = x + s\cos u$$

$$y' = y + s\sin u$$

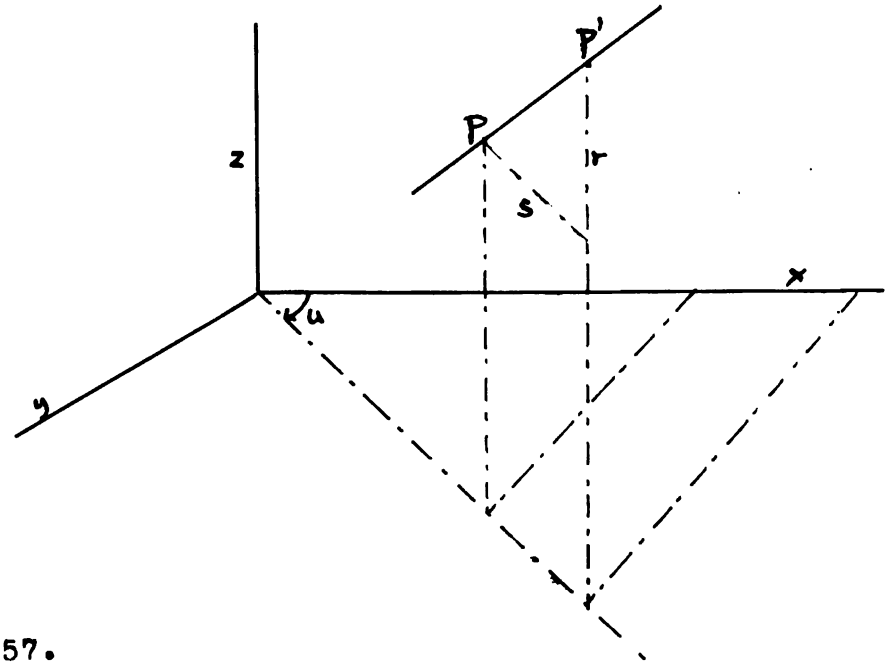

fig.57.

The equation of the curve becomes

$$x = (a + s)\cos u$$

$$y = (a + s)\sin u$$

$$z = au\,\mathrm{ctg}\,d + r.$$

We shall conclude the paper with the example above, but it is evident that the cissoidal constructions can produce not only a great variety of plane curves, but an equally great variety of surfaces, and an almost endless number of space curves.

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